Generalized Eigenvalue Decomposition in Time Domain Modal Parameter Identification

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This paper is intended to point out the relationship among current time domain modal analysis methods by employing generalized eigenvalue decomposition. Ibrahim time domain (ITD), least-squares complex exponential (LSCE) and eigensystem realization algorithm (ERA) methods are reviewed and chosen to do the comparison. Reformulation to their original forms shows these three methods can all be attributed to a generalized eigenvalue problem with different matrix pairs. With this general format, we can see that single-input multioutput (SIMO) methods can easily be extended to multi-input multioutput (MIMO) cases by taking advantage of a generalized Hankel matrix or a generalized Toeplitz matrix. [DOI: 10.1115/1.2775509]

1 Introduction

A fundamental problem in experimental modal analysis is to extract modal parameters of dynamical structures. These parameters include natural frequencies, damping ratios, and mode shapes. With this information, we can validate and update theoretical finite element models and gain a better understanding of corresponding structures. Another widely used application of these modal parameters is to monitor the health of a structure. During past decades, a vast range of modal parameter identification methods has been developed. These can be divided into two main categories: frequency domain and time domain methods. Frequency domain methods tend to provide more accurate results when the order of modes are relatively low, but they suffer specific problems associated with the fast Fourier transform (FFT) analysis, such as the leakage. In addition, frequency domain methods often need detailed input information to calculate frequency response functions. Time domain methods manipulate signals in time domain and usually need only output responses. This property makes time domain methods advantageous in practical tests. In addition, time domain methods are able to differentiate two modes that are close to each other. This paper is concerned with the time domain methods and is intended to provide a uniform formulation based on generalized eigenvalue decomposition. Based on the number of input and output channels used in tests, modal analysis algorithms can be classified as single-input single-output (SISO), e.g., the complex exponential (CE) method [1], single-input multioutput (SIMO) methods, e.g., the least-squares complex exponential (LSCE) method [2], Ibrahim time domain (ITD) method [3–6], and the multi-input multioutput (MIMO) methods (e.g., polyreference complex exponential (PRCE) method [7], eigensystem realization algorithm (ERA) [8,9], and stochastic subspace-based modal analysis methods [10–12]). CE, LSCE, and PRCE are all based on the Prony's theory and form a natural extension from SISO to SIMO and then to MIMO. The CE method [1] uses a square output matrix to estimate the coefficients of the Prony's polynomial and is sensitive to noise. LSCE was introduced by Brown et al. [2] in 1979. Its formulation is similar to the CE except that the displacement matrix is constructed by a multichannel output signal, and pseudoinverse technique is employed to estimate the coefficients of the Prony's polynomials. For both methods, a prior singular value decomposition [2] can significantly reduce the effect of noise. In 1982, Vold et al.[7] further extended LSCE to a MIMO case. The PRCE [7] method overcomes the problem associated with the SIMO method, when one of the modes may not be present in the output responses. Ibrahim has published a series of papers since 1973 and developed the so-called ITD method. The original paper started with the state space description and used the output displacement to reconstruct the governing equations of structures [3]. Later, a more general form was given, which was based on the explicit mathematical form of the output response [4]. In this new algorithm, the output signal can be displacement, velocity, or acceleration. The modal confidence factor [5] was also introduced to identify the real modes from the spurious computational modes. A detailed examination on the performance of ITD was given in Ref. [6]. Two more sophisticated time domain modal parameter identification algorithms are the eigensystem realization algorithm (ERA) [8] and stochastic subspace-based modal analysis methods [10–12]. Both were developed from control theory. Using a dynamical system in its state space form, ERA provides a minimum order realization of the system matrix by using free decay responses. ERA method also provides modal amplitude coherence and modal phase collinearity to quantitatively identify system and noisy modes. A detailed evaluation of the effect of noise on ERA was given in Ref. [9]. The stochastic subspace-based modal analysis uses the theory of stochastic subspace realization with an assumption that inputs are white noise. A good review of the popular modal parameter identification methods can be found in Ref. [1]. A unified matrix polynomial formulation of the current modal identification methods is given in Ref. [13].

In what follows, three popular time domain modal parameter identification methods—LSCE, ITD, and ERA—are chosen to explicitly show the underlying relationship among these methods. Original derivations of these methods are presented briefly and followed by reformulation based on generalized eigenvalue decomposition. This reformulation can be readily used to extend SISO modal analysis algorithms to SIMO and MIMO cases. The discussion and conclusion are given at the end.

2 Time Domain Modal Analysis Methods

2.1 Ibrahim Time Domain (ITD) Method. The ITD method is examined first to provide a uniform format by generalized eigenvalue decomposition. ITD was introduced in 1973 [3]. In that original paper, the authors used $2n$ state variables with a $2n$ length of time sequence, where n is the degree of freedom of a system, to

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identify the system matrix and modal parameters. This method relies on the exact format of system's governing equation, and the time signal should be clean displacement. In a later paper [4], a more general algorithm independent of the state space construction was proposed. The algorithm allows the analytical signals to be displacement, velocity, or acceleration measurements. The success of this updated algorithm relies on the fact that output signals have an exponentially decaying form.

The governing equations of motion for an n-degree-of-freedom free vibration system can be written as

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0} \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are $n \times n$ mass, damping, and stiffness matrices, $\ddot{\mathbf{x}}$, $\dot{\mathbf{x}}$, and $\mathbf{x}$ represent n-dimensional acceleration, velocity, and displacement vectors. The solution to these equations can be expressed in the form of

$$x_i(t) = \sum_{k=1}^{2n} \psi_{ik} e^{s_k t}, \quad i = 1, \ldots, n \quad \text{or} \quad x_i(t) = \boldsymbol{\psi}_i e^{\mathbf{s}t} \tag{2}$$

where $\boldsymbol{\psi}_i = [\psi_{i1}, \psi_{i2}, \ldots, \psi_{i2n}]$, $\mathbf{s} = [s_1, s_2, \ldots, s_{2n}]^T$, and t represents scalar time. $\boldsymbol{\psi}_i$ will constitute the desired mode shape matrix. For a lightly damped system, $s_k = -\xi_k \omega_{nk} + j\omega_{dk}$ provides the natural frequency and the damping ratio. ω_{nk}, ω_{dk}, and ξ_k represent kth undamped, damped natural frequencies, and damping ratio. j is the imaginary unit.

Suppose time responses from n sensors are sampled over a time period equal to $2n\Delta t$, where Δt is the sampling time interval. Then from Eq. (2), we have

$$\begin{bmatrix} x_1(t_0) & x_1(t_1) & \cdots & x_1(t_{2n-1}) \\ x_2(t_0) & x_2(t_1) & \cdots & x_2(t_{2n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_n(t_0) & x_n(t_1) & \cdots & x_n(t_{2n-1}) \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{12n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{22n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{n2n} \end{bmatrix} \begin{bmatrix} e^{s_1 t_0} & e^{s_1 t_1} & \cdots & e^{s_1 t_{2n-1}} \\ e^{s_2 t_0} & e^{s_2 t_1} & \cdots & e^{s_2 t_{2n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{s_{2n} t_0} & e^{s_{2n} t_1} & \cdots & e^{s_{2n} t_{2n-1}} \end{bmatrix}$$

or

$$\mathbf{X} = \tilde{\boldsymbol{\Psi}} e^{\mathbf{st}} \tag{3}$$

where $\mathbf{X} \in \mathbb{R}^{n \times 2n}$ is the trajectory matrix composed by n-channel output signals, $\tilde{\boldsymbol{\Psi}} = [\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \ldots, \boldsymbol{\psi}_n]^T$, and $\mathbf{t} = [t_0, t_1, \ldots, t_{2n-1}]$.

Then we consider the same time responses shifted by one sampling time interval (here one sampling time is chosen just for convenience)

$$\begin{bmatrix} x_1(t_1) & x_1(t_2) & \cdots & x_1(t_{2n}) \\ x_2(t_1) & x_2(t_2) & \cdots & x_2(t_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ x_n(t_1) & x_n(t_2) & \cdots & x_n(t_{2n}) \end{bmatrix} = \begin{bmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{12n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{22n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{n2n} \end{bmatrix} \boldsymbol{\Lambda} \begin{bmatrix} e^{s_1 t_0} & e^{s_1 t_1} & \cdots & e^{s_1 t_{2n-1}} \\ e^{s_2 t_0} & e^{s_2 t_1} & \cdots & e^{s_2 t_{2n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{s_{2n} t_0} & e^{s_{2n} t_1} & \cdots & e^{s_{2n} t_{2n-1}} \end{bmatrix}$$

or

$$\mathbf{X}_\tau = \tilde{\boldsymbol{\Psi}} \boldsymbol{\Lambda} e^{\mathbf{st}}, \quad \text{where} \quad \boldsymbol{\Lambda} = \begin{bmatrix} e^{s_1 \Delta t} & 0 & \cdots & 0 \\ 0 & e^{s_2 \Delta t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{s_{2n} \Delta t} \end{bmatrix} \tag{4}$$

$\mathbf{X}_\tau \in \mathbb{R}^{n \times 2n}$ is the one sampling time shifted trajectory matrix. $\boldsymbol{\Lambda} \in \mathbb{R}^{2n \times 2n}$ is a diagonal matrix with the entries equal to $\{e^{s_k \Delta t}\}_{k=1}^{2n}$.

Similarly, another time-shifted response matrix with respect to $\mathbf{X}_\tau$ can be formed

$$\mathbf{X}_{2\tau} = \tilde{\boldsymbol{\Psi}} \boldsymbol{\Lambda}^2 e^{\mathbf{st}} \tag{5}$$

Equations (3)–(5) can be combined as follows:

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{X}_\tau \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\Psi}} \\ \tilde{\boldsymbol{\Psi}} \boldsymbol{\Lambda} \end{bmatrix} e^{\mathbf{st}} \quad \text{or} \quad \bar{\mathbf{X}} = \boldsymbol{\Phi} e^{\mathbf{st}} \tag{6}$$

and

$$\begin{bmatrix} \mathbf{X}_\tau \\ \mathbf{X}_{2\tau} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\Psi}} \boldsymbol{\Lambda} \\ \tilde{\boldsymbol{\Psi}} \boldsymbol{\Lambda}^2 \end{bmatrix} e^{\mathbf{st}} \quad \text{or} \quad \hat{\mathbf{X}} = \boldsymbol{\Phi} \boldsymbol{\Lambda} e^{\mathbf{st}} \tag{7}$$

Equations (6) and (7) can be manipulated to eliminating $e^{\mathbf{st}}$ and get

$$\bar{\mathbf{A}} \boldsymbol{\Phi} = \boldsymbol{\Phi} \boldsymbol{\Lambda} \quad \text{and} \quad \bar{\mathbf{A}} = \hat{\mathbf{X}} \bar{\mathbf{X}}^{-1} \tag{8}$$

where $\bar{\mathbf{A}} \in \mathbb{R}^{2n \times 2n}$ is called the system matrix, containing the information of the dynamical system. $\boldsymbol{\Phi} \in \mathbb{R}^{2n \times 2n}$ is the complex mode shape matrix. Solving the eigenvalue problem of Eq. (8) provides us with all modal parameters. The above are the major steps in ITD analysis.

For the time periods longer than $2n$, Eq. (8) can be rewritten as

$$(\hat{\mathbf{X}} \bar{\mathbf{X}}^T)(\bar{\mathbf{X}} \bar{\mathbf{X}}^T)^{-1} \boldsymbol{\Phi} = \boldsymbol{\Phi} \boldsymbol{\Lambda} \tag{9}$$

More generally, we consider the joint diagnolization of matrix $\mathbf{R}_1$ and $\mathbf{R}_2$

$$\mathbf{R}_1 = \bar{\mathbf{X}} \bar{\mathbf{X}}^T = \boldsymbol{\Phi} e^{\mathbf{st}} (e^{\mathbf{st}})^T \boldsymbol{\Phi}^T$$

$$\mathbf{R}_2 = \bar{\mathbf{X}} \hat{\mathbf{X}}^T = \boldsymbol{\Phi} e^{\mathbf{st}} (e^{\mathbf{st}})^T \boldsymbol{\Lambda} \boldsymbol{\Phi}^T \tag{10}$$

Then, Eq. (9) can be rewritten using generalized eigenvalue decomposition as

$$\mathbf{R}_1 \boldsymbol{\Phi}^{-T} = \mathbf{R}_2 \boldsymbol{\Phi}^{-T} \boldsymbol{\Lambda}^{-1} \tag{11}$$

From Eq. (11), we can see the inverse and transpose of the generalized eigenvector matrix from matrix pair $(\mathbf{R}_1, \mathbf{R}_2)$ provides the complex mode shapes and the inverse of the generalized eigenvalues contains the natural frequencies and damping ratios.

In summary, the procedures for performing ITD by generalized eigenvalue decomposition are as follows:

- construct $\bar{\mathbf{X}}$ and $\hat{\mathbf{X}}$ as before
- assemble two matrices $\mathbf{R}_1 = \bar{\mathbf{X}} \bar{\mathbf{X}}^T$ and $\mathbf{R}_2 = \bar{\mathbf{X}} \hat{\mathbf{X}}^T$
- perform generalized eigenvalue decomposition to the matrix pair $(\mathbf{R}_1, \mathbf{R}_2)$
- extract modal parameters from the generalized eigenvalues and eigenvectors

It is noted that the smooth orthogonal decomposition (SOD) based modal analysis method [14] can be looked as another version of ITD, where the differentiated matrix instead the time shifted matrix is used. However, the SOD-based method can work directly with an n-dimensional data matrix.

2.2 Least-Squares Complex Exponential Method. Now ITD will be used as the baseline, and effort will be made to reformulate LSCE to a similar form. LSCE [2] and PRCE [7] constitute an extension of CE from SISO to SIMO and then the MIMO case. All of them are based on the Prony's theory. Here, LSCE is chosen to compare to ITD because both belong to the SIMO case. To better present the relationship, CE is discussed first.

CE, LSCE, and PRCE start with the impulse response function (IRF), which has the same form as Eq. (2). For CE, assuming the time responses are sampled at $2n+1$ stances, one gets

$$x_i(t_0)=\sum_{k=1}^{2n}\psi_{ik};\quad x_i(t_1)=\sum_{k=1}^{2n}\psi_{ik}e^{s_k\Delta t};\cdots;x_i(t_{2n})=\sum_{k=1}^{2n}\psi_{ik}e^{s_k 2n\Delta t} \tag{12}$$

or just

$$x_i(t_0)=\sum_{k=1}^{2n}\psi_{ik};\quad x_i(t_1)=\sum_{k=1}^{2n}\psi_{ik}V_k;\cdots;x_i(t_{2n})=\sum_{k=1}^{2n}\psi_{ik}V_k^{2n} \tag{13}$$

with $V_k=e^{s_k\Delta t}$.

To solve the unknowns V_k and ψ_{ik}, we use the Prony's method, which states if V_k appear in complex conjugate pairs then there exists a polynomial in V_k of order l (here $l=2n$) with real coefficients β, such that the following equation holds:

$$\beta_0+\beta_1V_k+\beta_2V_k^2+\cdots+\beta_{2n}V_k^{2n}=0 \tag{14}$$

Here, the solutions to V_k provide the modal parameters in complex conjugate pairs.

Then we multiply both sides of Eq. (13) with $\beta_0,\ldots,\beta_{2n}$ and sum them together

$$\sum_{j=0}^{2n}\beta_jx_i(t_j)=\sum_{j=0}^{2n}\left(\beta_j\sum_{k=1}^{2n}\psi_{ik}V_k^j\right)=\sum_{k=1}^{2n}\left(\psi_{ik}\sum_{j=0}^{2n}\beta_jV_k^j\right) \tag{15}$$

Substituting Eq. (15) into Eq. (14), we get

$$\sum_{j=0}^{2n}\beta_jx_i(t_j)=0 \tag{16}$$

To solve Eq. (16), we put a single time response x_i into the matrix form as follows:

$$\begin{bmatrix} x_i(t_0) & x_i(t_1) & \cdots & x_i(t_{2n}) \\ x_i(t_1) & x_i(t_2) & \cdots & x_i(t_{2n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_i(t_{2n-1}) & x_i(t_{2n}) & \cdots & x_i(t_{4n-1}) \end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_{2n}\end{bmatrix}=\mathbf{0} \tag{17}$$

Setting $\beta_{2n}=1$, Eq. (17) transforms to

$$\begin{bmatrix} x_i(t_0) & x_i(t_1) & \cdots & x_i(t_{2n-1}) \\ x_i(t_1) & x_i(t_2) & \cdots & x_i(t_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ x_i(t_{2n-1}) & x_i(t_{2n}) & \cdots & x_i(t_{4n-2}) \end{bmatrix}\begin{bmatrix}\beta_0\\ \beta_1\\ \vdots\\ \beta_{2n-1}\end{bmatrix}=-\begin{bmatrix}x_i(t_{2n})\\ x_i(t_{2n+1})\\ \vdots\\ x_i(t_{4n-1})\end{bmatrix} \tag{18}$$

or

$$\mathbf{X}_i\boldsymbol{\beta}=-\check{\mathbf{x}}_i \tag{19}$$

$\mathbf{X}_i\in\mathbb{R}^{2n\times 2n}$, $\check{\mathbf{x}}_i\in\mathbb{R}^{2n}$ and the subscript i means the matrices are constructed from a single output response. Equation (18) or Eq. (19) can be used to solve for $\boldsymbol{\beta}$. Then substituting $\boldsymbol{\beta}$ into Eq. (14) can lead to the roots of the polynomial and, therefore, the natural frequencies and damping ratios. The modal residues ψ_{ik} are solvable by rewriting Eq. (13) as

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ V_1 & V_2 & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_1^{2n-1} & V_2^{2n-1} & \cdots & V_{2n}^{2n-1} \end{bmatrix}\begin{bmatrix}\psi_{i1}\\ \psi_{i2}\\ \vdots\\ \psi_{i2n}\end{bmatrix}=\begin{bmatrix}x_i(t_0)\\ x_i(t_1)\\ \vdots\\ x_i(t_{2n-1})\end{bmatrix} \tag{20}$$

The mode shapes can be obtained by combining the modal residues calculated from different sampling sensors. The above are the basic ideas of CE.

The extension from CE to LSCE is quite straightforward. Instead of using a single output response, LSCE estimates the coefficients $\boldsymbol{\beta}$ in Eq. (14) by using several output time histories, and the mode shape matrix is obtainable in one step. For each output time history, we have equations similar to Eq. (18) or Eq. (19), then we assemble p output responses into one matrix as

$$\begin{bmatrix}\mathbf{X_1}\\ \mathbf{X_2}\\ \vdots\\ \mathbf{X_p}\end{bmatrix}[\boldsymbol{\beta}]=-\begin{bmatrix}\check{\mathbf{x}}_\mathbf{1}\\ \check{\mathbf{x}}_\mathbf{2}\\ \vdots\\ \check{\mathbf{x}}_\mathbf{p}\end{bmatrix} \tag{21}$$

$\boldsymbol{\beta}$ is then obtained by using pseudoinverse technique.

The mode shapes calculation follows the similar idea as CE, but using several output responses,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ V_1 & V_2 & \cdots & V_{2n} \\ V_1^2 & V_2^2 & \cdots & V_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ V_1^{2n-1} & V_2^{2n-1} & \cdots & V_{2n}^{2n-1} \end{bmatrix}\begin{bmatrix} \psi_{11} & \psi_{21} & \cdots & \psi_{2n1} \\ \psi_{12} & \psi_{22} & \cdots & \psi_{2n2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{12n} & \psi_{22n} & \cdots & \psi_{2n2n} \end{bmatrix}=\begin{bmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_{2n}(t_0) \\ x_1(t_1) & x_2(t_1) & \cdots & x_{2n}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t_{2n-1}) & x_2(t_{2n-1}) & \cdots & x_{2n}(t_{2n-1}) \end{bmatrix} \tag{22}$$

or in matrix format

$$\mathbf{V}\boldsymbol{\Phi}^T=\check{\mathbf{X}} \tag{23}$$

where

$$\mathbf{V}=\begin{bmatrix} 1 & 1 & \cdots & 1 \\ V_1 & V_2 & \cdots & V_{2n} \\ V_1^2 & V_2^2 & \cdots & V_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ V_1^{2n-1} & V_2^{2n-1} & \cdots & V_{2n}^{2n-1} \end{bmatrix}$$

$$\boldsymbol{\Phi}^T=\begin{bmatrix} \psi_{11} & \psi_{21} & \cdots & \psi_{2n1} \\ \psi_{12} & \psi_{22} & \cdots & \psi_{2n2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{12n} & \psi_{22n} & \cdots & \psi_{2n2n} \end{bmatrix}$$

and

$$\check{\mathbf{X}}=\begin{bmatrix} x_1(t_0) & x_2(t_0) & \cdots & x_{2n}(t_0) \\ x_1(t_1) & x_2(t_1) & \cdots & x_{2n}(t_1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(t_{2n-1}) & x_2(t_{2n-1}) & \cdots & x_{2n}(t_{2n-1}) \end{bmatrix}$$

Here, $\boldsymbol{\Phi}$ provides the complex mode shape matrix. Equations (21)–(23) are the fundamental steps of LSCE.

CE and LSCE use the Prony's technique to extract the modal parameters, whereas ITD achieves this by an eigenvalue decomposition or a generalized eigenvalue decomposition. At a first glance, they look different. However, further transform of CE and LSCE shows both methods are essentially equal to ITD.

First, let us consider the CE case. Equation (18) is used to calculate the coefficients β, which is used for solving the polynomial roots of Eq. (14). If we add more time instants to the right-hand side of Eq. (18) as

$$\begin{bmatrix}x_i(t_{2n})\\ x_i(t_{2n+1})\\ \vdots\\ x_i(t_{4n-1})\end{bmatrix}\rightarrow\begin{bmatrix} x_i(t_1) & x_i(t_2) & \cdots & x_i(t_{2n}) \\ x_i(t_2) & x_i(t_3) & \cdots & x_i(t_{2n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_i(t_{2n}) & x_i(t_{2n+1}) & \cdots & x_i(t_{4n-1}) \end{bmatrix} \tag{24}$$

then Eq. (18) becomes:

$$\mathbf{X}_i\,\text{ß} = \mathbf{X}_{i\tau} \quad \text{or} \quad \text{ß}^T\mathbf{X}_i = \mathbf{X}_{i\tau}, \tag{25}$$

where

$$\text{ß} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\beta_0 \\ 1 & 0 & \cdots & 0 & -\beta_1 \\ 0 & 1 & \cdots & 0 & -\beta_2 \\ \vdots & \vdots & \vdots & \ddots & -\beta_{2n-2} \\ 0 & 0 & \cdots & 1 & -\beta_{2n-1} \end{bmatrix} \tag{26}$$

$$\mathbf{X}_i = \begin{bmatrix} x_i(t_0) & x_i(t_1) & \cdots & x_i(t_{2n-1}) \\ x_i(t_1) & x_i(t_2) & \cdots & x_i(t_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ x_i(t_{2n-1}) & x_i(t_{2n}) & \cdots & x_i(t_{4n-2}) \end{bmatrix}$$

and

$$\mathbf{X}_{i\tau} = \begin{bmatrix} x_i(t_1) & x_i(t_2) & \cdots & x_i(t_{2n}) \\ x_i(t_2) & x_i(t_3) & \cdots & x_i(t_{2n+1}) \\ \vdots & \vdots & \ddots & \vdots \\ x_i(t_{2n}) & x_i(t_{2n+1}) & \cdots & x_i(t_{4n-1}) \end{bmatrix}$$

Here, ß or ß^T is called the *companion matrix* of the polynomial $p(u)$

$$p(u) = \beta_0 + \beta_1 u + \beta_2 u^2 + \cdots + \beta_{2n-1}u^{2n-1} + u^{2n} \tag{27}$$

where u is the polynomial variable. The roots of $p(u)$ can be found by calculating the eigenvalue decomposition of its *companion matrix*, and it is usually superior to the conventional root searching methods.

With *companion matrix* ß, the computation of Eq. (25) and then solving V_k can be accomplished in a single step by doing an eigenvalue decomposition

$$\text{ß}^T\mathbf{\Phi}_i = \mathbf{\Phi}_i\mathbf{\Lambda} \quad \text{or} \quad \mathbf{X}_{i\tau}\mathbf{X}_i^{-1}\mathbf{\Phi}_i = \mathbf{\Phi}_i\mathbf{\Lambda} \tag{28}$$

where $\mathbf{\Lambda}$ is a diagonal matrix with its entries as $e^{s_i\Delta t}$. The subscript i in $\mathbf{\Phi}_i$ is used to differentiate $\mathbf{\Phi}_i$ from mode shape matrix $\mathbf{\Phi}$ since this is a single time output case. As can be seen, Eq. (28) is quite similar to Eq. (8). Again for general output signals, Eq. (28) is expressed as

$$(\mathbf{X}_{i\tau}\mathbf{X}_i^T)(\mathbf{X}_i\mathbf{X}_i^T)^{-1}\mathbf{\Phi}_i = \mathbf{\Phi}_i\mathbf{\Lambda} \tag{29}$$

Written in a generalized eigenvalue decomposition form, Eq. (29) becomes

$$\mathbf{X}_i\mathbf{X}_i^T\mathbf{\Phi}_i^{-T} = \mathbf{X}_i\mathbf{X}_{i\tau}^T\mathbf{\Phi}_i^{-T}\mathbf{\Lambda}^{-1} \tag{30}$$

Equations (28) and (29) look quite similar to Eq. (9) except here matrix $\mathbf{X}_i$, $\mathbf{X}_{i\tau}$, and $\mathbf{\Phi}_i$ are composed of single output response.

To further examine the meaning of matrix $\mathbf{\Phi}_i^{-T}$, we consider Eq. (20) and add more time instants to the right-hand side. Equation (20) becomes

$$\bar{\mathbf{\Phi}}_i\mathbf{V}^T = \mathbf{X}_i \tag{31}$$

where

$$\bar{\mathbf{\Phi}}_i = \begin{bmatrix} \psi_{i1} & \psi_{i2} & \cdots & \psi_{i2n} \\ \psi_{i1}V_1 & \psi_{i2}V_2 & \cdots & \psi_{i2n}V_{2n} \\ \psi_{i1}V_1^2 & \psi_{i2}V_2^2 & \cdots & \psi_{i2n}V_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{i1}V_1^{2n-1} & \psi_{i2}V_2^{2n-1} & \cdots & \psi_{i2n}V_{2n}^{2n-1} \end{bmatrix} \tag{32}$$

and

$$\mathbf{V}^T = \begin{bmatrix} 1 & V_1 & \cdots & V_1^{2n-1} \\ 1 & V_2 & \cdots & V_2^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & V_{2n} & \cdots & V_{2n}^{2n-1} \end{bmatrix}$$

We also have

$$\begin{bmatrix} \psi_{i1}V_1 & \psi_{i2}V_2 & \cdots & \psi_{i2n}V_{2n} \\ \psi_{i1}V_1^2 & \psi_{i2}V_2^2 & \cdots & \psi_{i2n}V_{2n}^2 \\ \psi_{i1}V_1^3 & \psi_{i2}V_2^3 & \cdots & \psi_{i2n}V_{2n}^3 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{i1}V_1^{2n} & \psi_{i2}V_2^{2n} & \cdots & \psi_{i2n}V_{2n}^{2n} \end{bmatrix}\mathbf{V}^T = \mathbf{X}_{i\tau} \tag{33}$$

or

$$\bar{\mathbf{\Phi}}_i\mathbf{\Lambda}\mathbf{V}^T = \mathbf{X}_{i\tau} \quad \text{where} \quad \mathbf{\Lambda} = \begin{bmatrix} V_1 & 0 & 0 & \cdots & 0 \\ 0 & V_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & V_{2n} \end{bmatrix} \tag{34}$$

Equations (31) and (34) can be combined as

$$\mathbf{X}_{i\tau}\mathbf{X}_i^{-1}\bar{\mathbf{\Phi}}_i = \bar{\mathbf{\Phi}}_i\mathbf{\Lambda} \tag{35}$$

Examining Eqs. (35) and (28), we can see $\bar{\mathbf{\Phi}}_i = \mathbf{\Phi}_i$. From Eq. (32), we have

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ V_1 & V_2 & \cdots & V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ V_1^{2n-1} & V_2^{2n-1} & \cdots & V_{2n}^{2n-1} \end{bmatrix}\begin{bmatrix} \phi_{i1} & 0 & \cdots & 0 \\ 0 & \phi_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{i2n} \end{bmatrix} = \bar{\mathbf{\Phi}}_i \tag{36}$$

Therefore, $\mathbf{\Phi}_i$ is constructed by the multiplication of model residues with corresponding poles.

Now CE has been formulated into a generalized eigenvalue decomposition form, which is quite similar to ITD. However, ITD was developed initially for SIMO case and it is more appropriate to relate LSCE with ITD. In fact, when multioutput time histories are used, the relationship between LSCE and ITD is even clearer.

Starting with the Eq. (21), which aims at estimating the matrix ß^T by using several output responses, here for an n-degree-of-freedom system, we assume we have n output channels. Instead of shifting a single time output $2n-1$ times as CE, we can construct a matrix by assembling the original n-channel output history with its one sampling time shifted output history. Now, Eq. (25) becomes

$$[\text{ß}^T]\begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_2 \\ \mathbf{X}_3 \end{bmatrix} \quad \text{or} \quad [\text{ß}^T]\bar{\mathbf{X}} = \hat{\mathbf{X}} \tag{37}$$

Here we use different row order from the original LSCE since it will not affect the estimation of ß^T. From Eq. (37), we can estimate ß^T and calculate its eigenvalue decomposition to obtain modal parameters. This can be written in a generalized eigenvalue decomposition form

$$\bar{\mathbf{X}}\bar{\mathbf{X}}\mathbf{\Phi}^T = \bar{\mathbf{X}}\hat{\mathbf{X}}^T\mathbf{\Phi}^T\mathbf{\Lambda}^{-1} \tag{38}$$

which looks exactly the same as ITD.

2.3 Eigensystem Realization Algorithm. ERA was developed by Juang and Pappa in 1985 from system realization viewpoint [8]. It is a MIMO algorithm that can be used for modal parameter identification and model reduction of dynamical systems. The first step of this algorithm is to formulate a generalized *Hankel* matrix, which contains the *Markov* parameters. Then the realization matrices, which can reproduce system's input-output relationship, are derived. Modal parameters are extracted from the realized system matrices. The detailed derivation of ERA can be found in Ref. [8]. Here, only major steps of the algorithm are listed and used to construct a relationship with ITD.

ERA starts with the state-variable description of Eq. (1)

$$\mathbf{x}(j+1)=\mathbf{A}\mathbf{x}(j)+\mathbf{B}\mathbf{f}(j) \tag{39a}$$

$$\mathbf{y}(j)=\mathbf{C}\mathbf{x}(j) \tag{39b}$$

where $\mathbf{x}\in\mathbb{R}^{2n}$ is a state variable vector (assuming an n-degree-of-freedom system), $\mathbf{y}\in\mathbb{R}^{p}$ is an observation vector, $\mathbf{A}\in\mathbb{R}^{2n\times 2n}$ is a state transition matrix characterizing the dynamics of the system, $\mathbf{B}\in\mathbb{R}^{2n\times m}$ is an input matrix, $\mathbf{C}\in\mathbb{R}^{p\times 2n}$ is an output matrix, $\mathbf{f}\in\mathbb{R}^{m}$ is a control input vector, and j is the sampling index. It is assumed that the system is observed by n sensors and $\mathbf{y}$ has a dimension of $p=n$. The objective of ERA is to recover the three matrices $[\mathbf{A},\mathbf{B},\mathbf{C}]$, such that $\mathbf{y}$ is reproducible from Eq. (39).

For the free impulse response, the Markov parameters $\mathbf{Y}(j)$ are expressed (from control theory) as

$$\mathbf{Y}(j)=\mathbf{C}\mathbf{A}^{j-1}\mathbf{B} \tag{40}$$

where $\mathbf{Y}(j)$ has a dimension of $n\times l$ and l is the dimension of input channels. In practice, it is constructed by columnwise concatenation of observation vectors resulting from l different inputs

$$\mathbf{Y}(j)=[\mathbf{y}_1(j),\mathbf{y}_2(j),\ldots,\mathbf{y}_l(j)] \tag{41}$$

The ERA procedure can be summarized as follows:

Form a generalized Hankel matrix of dimension $rn\times sl$,

$$\mathbf{H}(j-1)=\begin{bmatrix}\mathbf{Y}(j) & \mathbf{Y}(j+1) & \cdot & \mathbf{Y}(j+s)\\ \mathbf{Y}(j+1) & \mathbf{Y}(j+2) & \cdot & \mathbf{Y}(j+s+1)\\ \vdots & \vdots & \ddots & \vdots\\ \mathbf{Y}(j+r) & \mathbf{Y}(j+r+1) & \cdot & \mathbf{Y}(j+r+s)\end{bmatrix} \tag{42}$$

where r and s are some kind of numbers, which need to be defined optimally,

Calculate singular value decomposition of the generalized Hankel matrix $\mathbf{H}(0)$

$$\mathbf{H}(0)=\mathbf{P}\mathbf{D}\mathbf{J}^T \tag{43}$$

Examine the nonzero floor of the singular values (usually it is just the dimension of the phase space: $2n$ in this case) and truncate the matrices $\mathbf{J}$ and $\mathbf{P}$ by keeping only their first $2n$ columns.

Construct $nr\times n$ $\mathbf{E}_n$ and $ls\times l$ $\mathbf{E}_l$ matrices

$$\mathbf{E}_n^T=[\mathbf{I}_n\ \mathbf{0}\ \ldots\ \mathbf{0}]$$

$$\mathbf{E}_l^T=[\mathbf{I}_l\ \mathbf{0}\ \ldots\ \mathbf{0}] \tag{44}$$

where $\mathbf{I}_l\in\mathbb{R}^{l\times l}$ and $\mathbf{I}_n\in\mathbb{R}^{n\times n}$ are identity matrices.

Estimate the realization matrices as

$$\mathbf{A}=\mathbf{D}^{-(1/2)}\mathbf{J}^T\mathbf{H}(1)\mathbf{J}\mathbf{D}^{-(1/2)},\quad \mathbf{B}=\mathbf{D}^{1/2}\mathbf{J}^T\mathbf{E}_l,\quad \text{and}\quad \mathbf{C}=\mathbf{E}_n^T\mathbf{P}\mathbf{D}^{1/2} \tag{45}$$

Calculate all modal parameters using the estimated system matrices $\mathbf{A}$ and $\mathbf{C}$.

As a result of the above procedure, the eigenvalues of $\mathbf{A}$ provide natural frequencies and damping ratios. Assume the eigenvector matrix of $\mathbf{A}$ is $\mathbf{\Theta}$, then the mode shape matrix is obtained using following transformation:

$$\mathbf{\Phi}=\mathbf{C}\mathbf{\Theta}=\mathbf{E}_n^T\mathbf{P}\mathbf{D}^{1/2}\mathbf{\Theta} \tag{46}$$

Equation (45) constitutes a minimum realization for the given system. After examining the singular values of $\mathbf{H}(0)$ and truncating $\mathbf{P}$ and $\mathbf{J}$ matrices, $\mathbf{A}=\mathbf{D}^{-(1/2)}\mathbf{P}^T\mathbf{H}(1)\mathbf{J}\mathbf{D}^{1/2}$ has a rank of $2n$.

From control theory, a nonsingular linear transformation $\mathbf{T}$ of the state vector $\mathbf{x}(j)$ leads to a transformed realization triple, such as $[\mathbf{T}\mathbf{A}\mathbf{T}^{-1},\mathbf{T}\mathbf{B},\mathbf{C}\mathbf{T}^{-1}]$, which is also a realization of the given dynamical system. $\mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ is called similarity transformation, which will not change the eigenvalues of the matrix $\mathbf{A}$.

To relate ERA to ITD, matrices $\mathbf{H}(0)$ and $\mathbf{H}(1)$ are considered after the singular value decomposition, which means both have the number of rows equal to the dimension of the system (here $2n$). Using a nonsingular matrix $\mathbf{T}=\mathbf{P}\mathbf{D}^{1/2}$, the similarity transformation of $\mathbf{A}$ is given by

$$\begin{aligned}\mathbf{T}\mathbf{A}\mathbf{T}^{-1}&=\mathbf{P}\mathbf{D}^{1/2}\mathbf{D}^{-(1/2)}\mathbf{P}^T\mathbf{H}(1)\mathbf{J}\mathbf{D}^{-(1/2)}\mathbf{D}^{-(1/2)}\mathbf{P}^{-1}\\&=\mathbf{P}\mathbf{P}^T\mathbf{H}(\mathbf{1})\mathbf{J}\mathbf{D}^{-1}\mathbf{P}^{-1}=\mathbf{H}(1)\mathbf{H}(0)^{\sharp}\end{aligned} \tag{47}$$

where $\sharp$ denotes a pseudoinverse, $\mathbf{P}$ and $\mathbf{J}$ are unitary matrices from singular value decomposition as mentioned above. Thus, $\mathbf{P}^{-1}=\mathbf{P}^T$ and $\mathbf{J}^{\sharp}=\mathbf{J}^{-T}$.

Natural frequencies and damping ratios can be extracted from the eigenvalue decomposition of the transformed matrix $\mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ or $\mathbf{H}(1)\mathbf{H}(0)^{\sharp}$. To identify mode shapes, we first look at the matrix $\mathbf{C}$ multiplied by $\mathbf{T}^{-1}$:

$$\mathbf{C}\mathbf{T}^{-1}=\mathbf{E}_n^T\mathbf{P}\mathbf{D}^{1/2}\mathbf{D}^{-(1/2)}\mathbf{P}^T=\mathbf{E}_n^T \tag{48}$$

Therefore, from Eqs. (47) and (48), we know the eigenvector matrix of $\mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ or $\mathbf{H}(1)\mathbf{H}(0)^{\sharp}$ directly provides mode shapes.

In summary, ERA in the SIMO case can be formulated as the following eigenvalue problem:

$$\mathbf{H}(1)\mathbf{H}(0)^{\sharp}\mathbf{\Phi}=\mathbf{\Phi}\mathbf{\Lambda} \tag{49}$$

or, in a generalized eigenvalue decomposition form,

$$\mathbf{H}(0)\mathbf{H}(0)^T\mathbf{\Phi}^{-T}=\mathbf{H}(0)\mathbf{H}(1)^T\mathbf{\Phi}^{-T}\mathbf{\Lambda}^{-1} \tag{50}$$

Equation (50) shows that ERA can also be realized by generalized eigenvalue decomposition, which is quite similar to the ones used for ITD or LSCE.

3 Discussion and Conclusion

In addition to ERA, stochastic subspace-based modal parameter extraction methods [10–12] are also developed from a control theory viewpoint. Similar to ERA, stochastic subspace methods aim to identify state space model Eq. (39) and modal parameters using only the output response. These methods assume the input is a stochastic process or white noise, while ERA is based on free impulse responses. Although the generalized Hankel matrix is used in ERA, stochastic subspace-based methods start with the generalized Toeplitz matrix. Various stochastic subspace-based modal parameter extraction algorithms exist, such as a data-driven–based algorithm [11], covariance-based algorithm [10], etc. The other steps in the stochastic subspace modal parameter extraction procedures are quite similar to the ERA, described in detail in Sec. 2.3. Thus, if in the ERA derivation the Hankel matrix is replaced by the Toeplitz matrix, the stochastic subspace methods can also be reformulated into a generalized eigenvalue problem.

In summary, this paper examines current popular time domain modal analysis methods and explicitly reformulates them by use of generalized eigenvalue decomposition. As shown in the above discussion, ITD, LSCE, and ERA are essentially similar if formulated by eigenvalue decomposition or, more generally, generalized eigenvalue decomposition. With this uniform form, we can see the matrix $\mathbf{X}_i$ in CE can be replaced by $\bar{\mathbf{X}}$ as in ITD, therefore generalizing CE to the SIMO case. Also in ITD, $\bar{\mathbf{X}}$ can be replaced by $\mathbf{H}$ as in ERA, and this generalizes ITD from SIMO to the MIMO case. In addition, the generalized eigenvalue decomposition formalism makes it clear that the crucial point for extracting modal parameters in time domain analysis is to simultaneously diagonalize two matrices, one of which is the correlation matrix of the output signal itself, and the other is the cross-correlation matrix with its time-shifted output signal or its differentiated matrix as Ref. [14]. More importantly, this simultaneous diagonalization makes the time domain modal analysis methods similar to some algorithms in BSS [15,16] since both employ this simultaneous diagonalization procedure [17].

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