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Reconstructing slow-time dynamics from fast-time measurements

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This paper considers a dynamical system subjected to damage evolution in variable operating conditions to illustrate the reconstruction of slow-time (damage) dynamics using fast-time (vibration) measurements. Working in the reconstructed fast-time phase space, phase space warping-based feature vectors are constructed for slow-time damage identification. A subspace of the feature space corresponding to the changes in the operating conditions is identified by applying smooth orthogonal decomposition (SOD) to the initial set of feature vectors. Damage trajectory is then reconstructed by applying SOD to the feature subspace not related to the changes in the operating conditions. The theory is validated experimentally using a vibrating beam, with a variable nonlinear potential field, subjected to fatigue damage. It is shown that the changes in the operating condition (or the potential field) can be successfully separated from the changes caused by damage (or fatigue) accumulation and SOD can identify the slow-time damage trajectory.

Keywords: multidimensional damage identification; slow-time dynamics reconstruction; variable operating conditions

1. Introduction

The study of underlying slow-time processes using readily available fast-time measurements is an important subject of many contemporary research efforts. The fields of structural health monitoring, damage diagnosis and prognosis, early disease detection and tracking are just a few illustrative examples of these activities. All of these share a similar basic mechanism, where hidden and slowly evolving process leads to deterioration in the system's fast-time performance and even to complete failure or catastrophe. It is also usually possible to measure fast-time dynamic behaviour, or response, of these systems using standard sensors and procedures. Then, the task is to use these measurements to infer the state of the underlying slow process and predict its future evolution.

The ultimate goal is to use fast-time measurements for predicting underlying slow-time process evolution, so that imminent failures are anticipated and preventive measures can be taken. For a meaningful prediction, both the estimates of the current state of slow-time process and its accurate model are

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One contribution of 8 to a Theme Issue 'Experimental nonlinear dynamics I. Solids'.

needed. However, for many practical applications these needs are hard to satisfy. For example, in fatigue damage accumulation there is no clear understanding of what an appropriate observable damage state should be, or what the corresponding physical model is (Bolotin 1998). Therefore, there is a clear need for technology that would provide tools for determining both appropriate observable slow-time state variables and their time evolution.

In this paper, a general framework, based on the dynamical systems perspective, for reconstructing underlying slow-time process dynamics is described. This approach naturally integrates the ability to identify the number of observable slow-time state variables and their time evolution in an appropriate phase space. Therefore, it provides needed information for constructing or validating slow-time predictive models, which can directly use available fast-time measurements. In previous works, it was demonstrated that, in a stationary operating environment, this approach can be successful for identifying a fatigue damage (Chelidze & Cusumano 2004, 2006; Chelidze & Liu 2005) or for multidimensional drift tracking (Chelidze 2004; Chelidze & Liu 2006). In both of these cases, *phase space warping* (PSW)-based feature vectors were analysed using *smooth orthogonal decomposition* (SOD) to identify the slow processes.

The focus of this study is on the identification of important slow-time processes, which are accompanied by unrelated operational changes that possess approximately the same time scales. As a paradigm for this problem, a dynamical system subjected to damage evolution in variable operating conditions is considered. The important process to identify is damage accumulation. The problem with the previously reported approach is that it will identify *all* slow-time changes but cannot differentiate between the changes in operating conditions and damage if they both possess the same time scales. Thus, the goal of this study is to develop a technique that will be insensitive to the changes in operating conditions and will only identify damage processes.

To illustrate and validate the proposed technique, an experimental apparatus used in previous studies was modified to introduce a two-dimensional change in the operating conditions and a material fatigue process. The experimental data are used to test a hypothesis that *changes in operating conditions and damage occupy separate subspaces in the PSW-based feature space, and these subspaces can be uniquely identified using the SOD*.

The structure of this paper is as follows. In §2, a brief overview of related work is provided. Next, in §3, a general framework and algorithm for reconstructing slow-time dynamics are described. This is followed in §4 by a description of an experimental system used to validate the framework. Section 5 describes the experimental results, which is followed by discussion of these results (§6) and conclusions (§7).

2. Background

The embedding theorem and phase space reconstruction through delay coordinate embedding (Sauer *et al.* 1991) have provided a fertile ground for the development of many algorithms for studying nonlinear dynamical systems. However, in this development, one of the main assumptions is that these systems are stationary (their parameters are fixed in time). In practice, this requirement is relaxed a little since there are always some parameter variations. It is usually assumed that these parameters drift slowly and their variation over the time period of observation is negligible. This assumption of quasi-stationarity can be circumvented if the dynamical model is extended to include also the dynamics of parameter variations. This paper considers exactly these types of systems that possess coupling between the fast-time directly observable or measurable dynamics and slow-time hidden drift dynamics.

To reconstruct the full extended phase space of systems considered here, data should be collected over a very long time span to capture slow-time variations, but at the same time should be sampled fast so that fast-time dynamics is also adequately described. This results in prohibitively large datasets that need to be embedded in high-dimensional phase spaces, which is a difficult problem from a practical viewpoint. Therefore, the conventional approach is to collect data over intermediate time scales that capture just fast-time dynamics ignoring small variations in parameters. Then significant changes in parameters can be detected by observing the changes in the characteristics of fast-time dynamics. Therefore, the focus of this paper is on the reconstruction of slow-time drift dynamics by characterizing the changes in the fast-time dynamics.

The structural health monitoring field works in the same data collection framework: feature vectors are extracted and analysed from data collected over intermediate time scales. Previous works (Chelidze 2004; Chelidze & Liu 2005, 2006; Chelidze & Cusumano 2006) include detailed reviews of the corresponding technologies. Some of the recent developments are in the area of damage diagnosis under time-varying environmental conditions. One approach is to measure all independent environmental factors, not a very practical option, and perform correlation analysis between the fast-time vibration features and environmental conditions (e.g. Peeters *et al.* 2001 and references within). Another approach focuses on identifying feature subspaces relating to the independent environmental factors (Yan *et al.* 2005*a*,*b*) without requiring to measure all these factors. In this formulation, the assumption is that the damage-related changes and the environmental changes occupy mutually orthogonal subspaces in the features space for the linear systems and can be identified using *principal component analysis* (PCA). In the nonlinear case, the local PCA analysis is used to effectively perform a piecewise linearization of a nonlinear problem. This subspace identification idea is also applied to the Henkel matrix formed from vibration measurements (Yan & Golinval 2006). The earlier work (Sohn & Farrar 2001) develops an ensemble of datasets for the undamaged system in various environmental conditions and compares features for the damaged system with 'similar' ones from this ensemble using linear model residual analysis. This work is further advanced in Sohn *et al.* (2002), where an autoassociative neural network is used to differentiate between the effects of damage and operating conditions.

The developments described above are mainly aimed at *detecting* damage in variable environmental or operating conditions. Some of them have the ability to locate damage or assess the severity or type of damage. However, these methods do not provide a comprehensive framework that can be used to determine both the dimensionality (number of independent modes) of damage and their time evolution in damage or slow-time phase space.

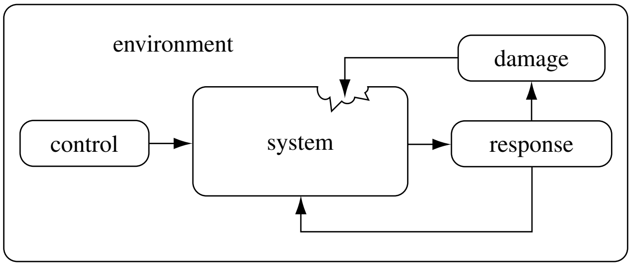

Figure 1. A paradigm for the class of systems considered.

The idea of reconstructing the parameter space of a dynamical system has been advocated independently by Güttler *et al.* (2001). Using numerical experiments, it was demonstrated that the smooth variation in d parameters can be reconstructed as a d-dimensional manifold in a high-dimensional feature space (composed of standard long-time statistics). It was stipulated that complications arising from bifurcations in deterministic dynamics are not significant when some amount of noise is present in the data. However, earlier analysis has indicated that these bifurcations are actually important considerations, since they destroy the needed one-to-one smooth relationship between the feature vectors and the slowly drifting parameters (Chelidze 2000; Chelidze *et al.* 2002). Thus, more careful attention should be paid to the choice of feature vectors. Another work (Verdes *et al.* 2001, 2006) has focused on the investigation of non-stationary time series and reconstruction of scalar driving force. Initially, the evolution of the external force causing the non-stationarity was traced by locally linearizing the dynamics with the shifting parameter (Verdes *et al.* 2001). Later, an overembedding method for modelling non-stationarity was investigated, where a slow scalar driving signal is estimated simultaneously with the intrinsic stationary dynamics (Verdes *et al.* 2006). Both of these methods address the reconstruction of scalar non-stationary dynamics and it is not clear as to how they can be used to address the multivariate problems.

The success of our approach depends on three main factors. First, fast-time data-based feature vectors should provide one-to-one smooth mapping to actual damage states. Second, one needs a method that can identify these damage states directly from the feature vectors, without *a priori* knowledge of the mapping. Finally, one needs an ability to separate variabilities caused by damage and changes in the environmental and operating conditions. Much of the previous work (Chelidze 2004; Chelidze & Liu 2006) was devoted to addressing the first two requirements by developing PSW-based feature vectors and the SOD-based damage identification. This paper focuses on the final requirement.

3. Theory

In §3*a*, a simple paradigm used in this study and associated general reasoning for what is needed for reconstructing a slow-time trajectory are described. In §3*b*, a practical algorithm used to analyse the experimental data is outlined.

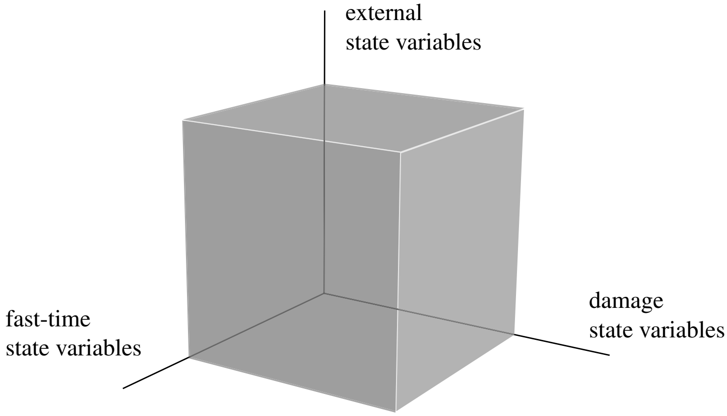

Figure 2. An extended phase space for the paradigm.

(*a*) *A paradigm: damage evolution in hierarchical dynamical system*

As a paradigm for a class of systems considered here, we consider a general dynamical system with slowly evolving damage as depicted in figure 1. This system can be described by the following equations that show coupling between the fast- and slow-time variables:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu}(\boldsymbol{\phi}), t), \qquad \dot{\phi} = \epsilon \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{\phi}, t) \quad \text{and} \quad y = h(\boldsymbol{x}), \tag{3.1}$$

where $\boldsymbol{x}$ is a fast-time directly observable or measurable variable; $\boldsymbol{\phi}$ is a hidden, slow-time damage variable; $\boldsymbol{\mu}$ represents a vector of parameters that are functions of $\boldsymbol{\phi}$; ϵ is a small rate constant describing the time-scale separation; t corresponds to time and describes all external time-varying inputs such as excitation or operating conditions; and y represents scalar measurement taken from the fast-time system through some scalar measurement function h. Thus, the goal is to use the measurements y to reconstruct the time evolution of ϕ.

The full extended phase space of the system of equations (3.1) can be visualized as shown in figure 2. The coordinates of this space include all external time-varying factors (excitation, operational and environmental variables) as well as the collection of all fast- and slow-time state variables. We assume that this phase space is bounded in fast-time and external variables. We also assume that one of the slow-time damage variables is monotonic in time. Then, data collections over intermediate time scales of the system can be thought to occur in a thin slice perpendicular to the slow-time axis in figure 2, as shown in figure 3*a*. Assuming that changes in the slow-time (i.e. external and damage) variables are negligible during each observation block, we can picture these as observation snapshots taken at a particular slow-time damage state as shown in figure 3*b*.

For each observation snapshot, we collect measurements from the fast-time system. These measurements can be a collection of multiple sensor outputs measuring both input and output variables. Measurements of system loads and environmental conditions would be of special importance in variable operating conditions (Peeters *et al.* 2001). However, here it is assumed that only one scalar measurement of a system's fast-time response is available. It is also assumed that this measurement captures dynamics that is sensitive to the slow damage evolution. Given only this measurement, one has to infer the slow-time damage

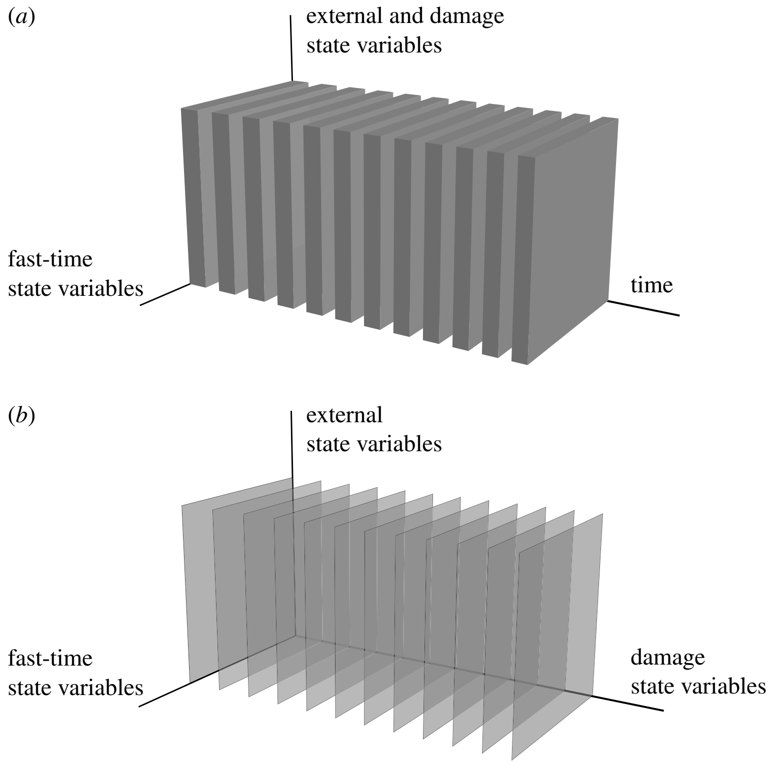

Figure 3. (*a*) Observation time blocks and (*b*) observation snapshots used in data collection.

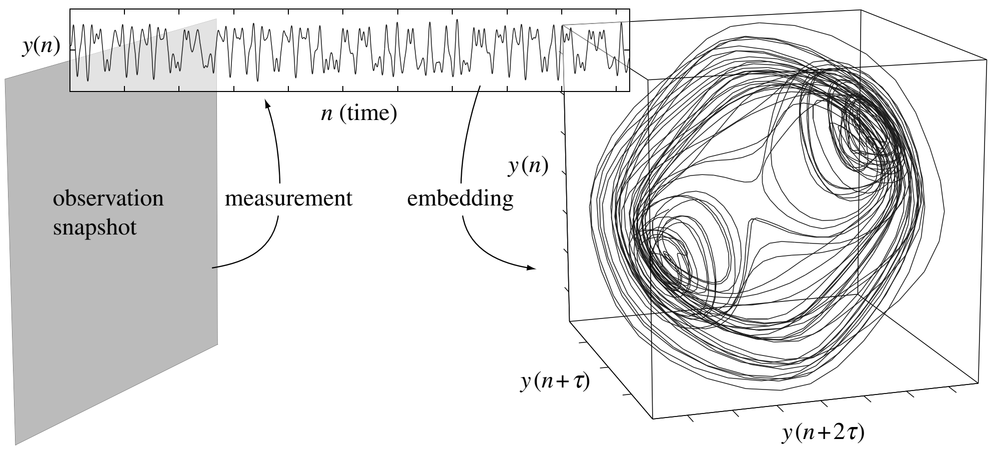

Figure 4. Schematics of phase space reconstruction for fast-time dynamics.

dynamics. The most general way of doing this would be to estimate fast-time model parameters, if such a model is available. Usually, these models are not available or are hard to identify. Thus, we focus on reconstructing a fast-time phase space through delay coordinate embedding and investigate changes in this space that are due to damage evolution. This process is depicted schematically in figure 4.

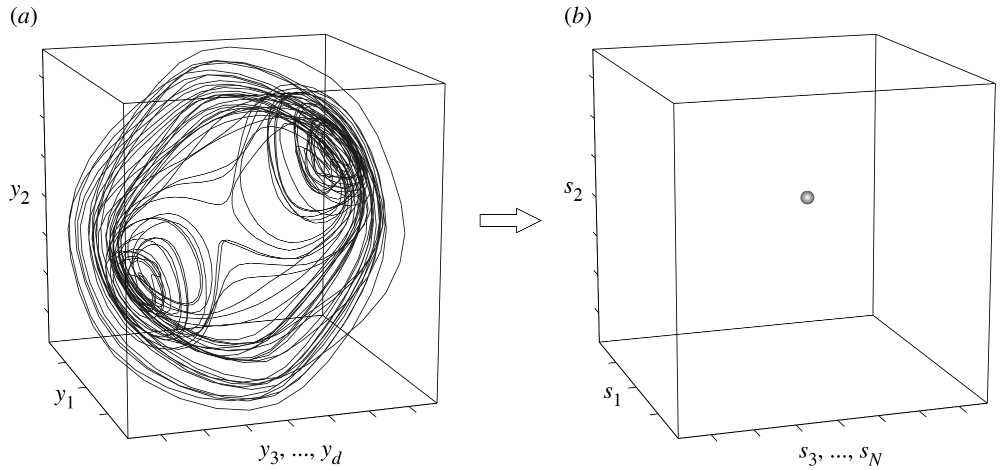

Figure 5. Projecting the (*a*) reconstructed fast-time phase space data onto the (*b*) feature space.

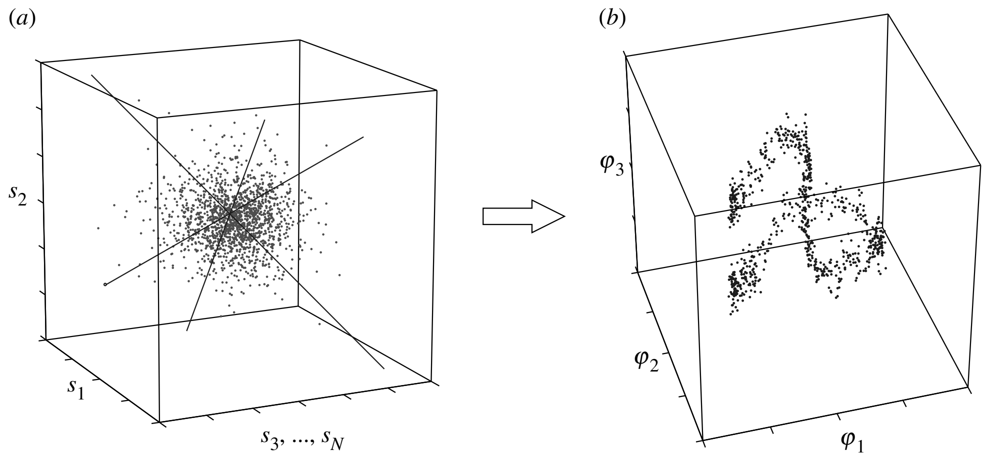

Figure 6. Projecting the (*a*) feature space onto the (*b*) slow-time phase space.

Working in the fast-time phase space, one needs to develop metrics that are sensitive to the changes in dynamics that are caused by damage. This process is usually called the development of feature vectors or feature extraction. Most of the features discussed in the literature are not only sensitive to damage, but also to the qualitative changes (e.g. bifurcations) in fast-time dynamics. This will suffice if the aim is to just detect the change in parameters. However, for the reconstruction of slow-time dynamics, features that are in a smooth, one-to-one relationship with damage states are necessary. For each observation snapshot, one feature vector is calculated; this projects the fast-time phase space data onto one point in the feature space as shown in figure 5. Therefore, the collection of observations would yield a collection of points in the feature space.

If the features were sensitive to damage, these points should contain the needed information for reconstructing slow-time phase space. However, the situation is complicated by the fact that these metrics can also be sensitive to the changes in external (i.e. operating and environmental) variables. Thus, one needs to separate slow-time damage variations from external variations. For small changes in damage, this separation can be accomplished by identifying linear subspaces in the feature space which are sensitive to only changes in damage as shown in figure 6. The hypothesis is that in this projection we will reconstruct a slow-time damage trajectory.

For the above strategy to work, the damage variability and external variability effects need to occupy different subspaces in the feature space. The problem will not be tractable if these subspaces are exactly the same. However, if damage manifests itself in a subspace not affected by the external variations, the methodology described here should generically work, provided that damage is unfolded in that subspace. The degree of success will be mainly determined by the degree of unfolding in the feature space. For example, if we have only scalar feature space, we cannot identify separate subspaces. Therefore, feature space should be high dimensional enough to allow unfolding of damage variations.

(*b*) *Algorithm: PSW and SOD*

Scalar data $\{y(n)\}_{n=0}^{M}$ are collected over consecutive intermediate time intervals of $t_{\mathrm{D}}=Mt_{\mathrm{s}}$ length, where t_{s} is the sampling time period dictated by the fast-time scale. The size of each data record described by M should be large enough to adequately describe fast-time dynamics. Then, the fast-time phase space points are reconstructed using delay coordinate embedding as $\boldsymbol{y}_n=[y(n), y(n+\tau), \ldots, y(n+(d-1)\tau)]^{\mathrm{T}} \in \mathbb{R}^d$, where τ is the delay time, d is the embedding dimension and T is used to denote transpose. The embedding parameters are determined using conventional techniques discussed in detail in Kantz & Schreiber (2004).

In the reconstructed fast-time phase space, one needs a metric that is in one-to-one smooth relationship with damage variables. Such a metric, which measures the difference in short-time evolutions for current and reference damage states, was first developed in Chelidze (2000) and Chelidze *et al.* (2002) and further extended to multidimensional form in Chelidze (2004). This metric can be viewed to provide a measure of deformation or warping of fast-time phase space trajectories due to slow-time drifts. A detailed discussion is given in Chelidze & Cusumano (2006), which advocates advantages of this type of shot-time statistics over conventional long-time statistics that are susceptible to corruption by bifurcation noise. Here, bifurcation noise refers to the changes in features caused by the qualitative changes in the fast-time response of the system. The particular metric used in this paper is called the *PSW function* (PSWF) and is defined as

$$\boldsymbol{e}_{\mathrm{R}}(\boldsymbol{\phi}; \boldsymbol{y}_n) \triangleq \boldsymbol{y}_{n+1} - \boldsymbol{P}(\boldsymbol{y}_n; \boldsymbol{\phi}_{\mathrm{R}}), \tag{3.2}$$

where $\boldsymbol{y}_n$ is a point in the reconstructed fast-time phase space for the current damage state $\boldsymbol{\phi}$; $\boldsymbol{y}_{n+1}$ is its image one time step later; and $\boldsymbol{P}(\boldsymbol{y}_n; \boldsymbol{\phi}_{\mathrm{R}}) \triangleq \boldsymbol{y}_{n+1}^{\mathrm{R}}$ is the image of $\boldsymbol{y}_n$ for a reference or healthy damage state $\boldsymbol{\phi}_{\mathrm{R}}$.

In Chelidze (2004), for developing multidimensional tracking vector, it was proposed to partition phase space into disjoint small hypercuboids ($\mathcal{B}_i$, $i=1, \ldots, N_d$) and evaluate the expected value of the PSWF in each of these regions,

$$\boldsymbol{e}_i(\boldsymbol{\phi}) = \frac{1}{N_i} \sum_{\boldsymbol{y} \in \mathcal{B}_i} \hat{e}(\boldsymbol{\phi}; \boldsymbol{y}), \tag{3.3}$$

where N_i is the number of points in $\mathcal{B}_i$ and $\hat{\boldsymbol{e}}$ is an estimate of the PSWF, using some data-based model for $\boldsymbol{P}(\boldsymbol{y}, \boldsymbol{\phi}_{\mathrm{R}})$. All these averaged PSWFs are assembled into $m = dN_{\mathrm{d}}$-dimensional feature vector for each observation snapshot $\{j\}_{j=1}^{q}$,

$$\boldsymbol{e}^j = [\boldsymbol{e}_1(\boldsymbol{\phi}); \boldsymbol{e}_2(\boldsymbol{\phi}); \ldots; \boldsymbol{e}_{N_{\mathrm{d}}}(\boldsymbol{\phi})], \tag{3.4}$$

where semicolons indicate column-wise concatenation of each $\boldsymbol{e}_i$.

In Chelidze (2004), it was also demonstrated that these feature vectors can be projected onto actual damage states if the total change in the damage state was small. To check if the damage phase space can be reconstructed without the *a priori* knowledge of this projection, the feature vectors were stacked in a time sequence as row vectors into a *tracking matrix* $\boldsymbol{Y} \in \mathbb{R}^{q \times m}$. It was then established that SOD of this tracking matrix provided the reconstructed damage phase space that was in affine relationship with the actual phase space.

The SOD is performed using generalized singular value decomposition of matrix $\boldsymbol{Y}$ and its time derivative $\boldsymbol{DY}$, where $\boldsymbol{D}$ is a discrete differential operator,

$$\boldsymbol{Y} = \boldsymbol{UCX}^{\mathrm{T}} \quad \text{and} \quad \boldsymbol{DY} = \boldsymbol{VSX}^{\mathrm{T}}, \tag{3.5}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are unitary matrices; $\boldsymbol{C}$ and $\boldsymbol{S}$ are diagonal matrices; and $\boldsymbol{X}$ is a square matrix. The *smooth orthogonal coordinates* (SOCs) are given by the columns of $\boldsymbol{\varphi} = \boldsymbol{UC}$, *smooth orthogonal modes* (SOMs) are provided by the columns of $\boldsymbol{\Psi} = \boldsymbol{X}^{-\mathrm{T}}$ and *smooth orthogonal values* (SOVs) are $\boldsymbol{\sigma} = \mathrm{diag}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C})/\mathrm{diag}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{S})$. The dominant SOVs carry information about the dimensionality of damage phase space and the corresponding SOCs reconstruct that phase space.

This algorithm was successfully applied to the tracking and prediction of fatigue damage evolution (Chelidze & Liu 2005; Chelidze & Cusumano 2006), where it was shown to be characterized by a scalar damage variable. It was also demonstrated that the algorithm can identify two-dimensional damage trajectory's phase space (Chelidze 2004; Chelidze & Liu 2005). In addition to the changes in the slow-time variables, these systems were observed to go through bifurcations in their fast-time behaviour. The SOD was shown to be a powerful tool for eliminating the changes caused by these abrupt transitions in the system dynamics and focusing on the gradual damage evolution. However, all these systems were tested in a stationary forcing environment (harmonic input forcing), and no operational and environmental changes were present. In general, one needs to consider these changes, which can also be characterized by time scales similar to damage time scale.

In this treatment, we show how the technology described above can be extended to systems in variable operating conditions. The fundamental assumption is that the systems of interest can be observed in variable external conditions while they are not subjected to actual damage evolution or during their healthy operation. In this way, the identification of the feature subspace that is affected by the changes in operating or environmental conditions is possible, and damage identification is then conducted in the feature subspace that is orthogonal to the operational subspace.

The procedure for identifying damage in variable external conditions can be summarized as follows.

(i) Assemble a tracking matrix $\tilde{\boldsymbol{Y}}$, which includes only the changes in external conditions (i.e. only using data recorded during healthy stage of operation).

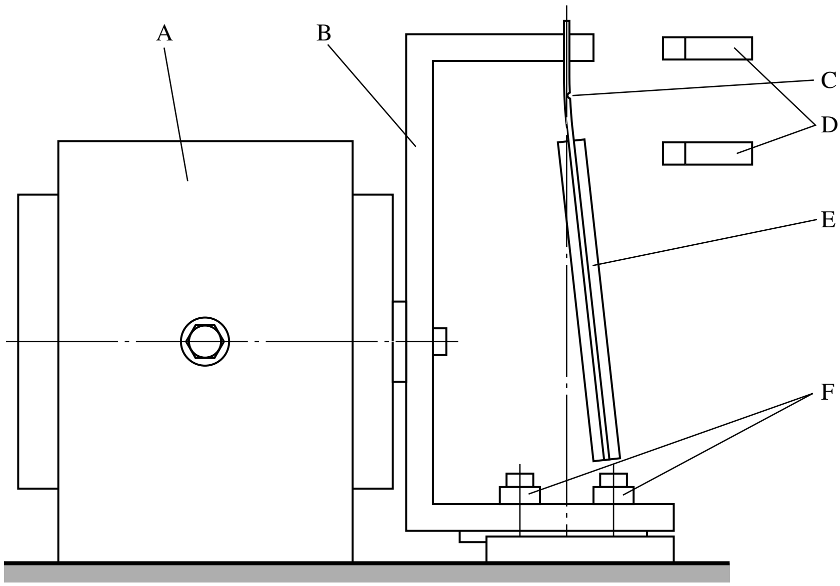

Figure 7. Schematic of the experimental apparatus: A, electromagnetic shaker; B, frame; C, notch in the beam; D, laser vibrometers; E, steel beam with stiffeners; F, permanent/electromagnet stacks.

(ii) Identify v largest SOVs of $\tilde{\boldsymbol{Y}}$, which are related to only external or operational variations, and the corresponding SOMs $\tilde{\psi}_k$ ($k=1, \ldots, v$).

(iii) Assemble another tracking matrix $\boldsymbol{Y}$ using all the recorded data, and for further analysis use only the subspace span by other SOMs $\tilde{\psi}_k$ ($k=v+1, \ldots, N_{\mathrm{s}}$).

(iv) Identify damage through SOD of truncated tracking matrix $\bar{\boldsymbol{Y}}=\boldsymbol{Y}\bar{\boldsymbol{\Psi}}$, where $\bar{\boldsymbol{\Psi}}=[\tilde{\psi}_{v+1}, \ldots, \tilde{\psi}_{N_{\mathrm{s}}}]$.

4. Experiment

The experimental system is a modified version of a well-known, two-well magnetoelastic oscillator (Moon & Holms 1979), earlier versions of which were also employed in previous studies (Chelidze *et al.* 2002; Chelidze & Cusumano 2004, 2006; Chelidze & Liu 2005). The schematic of the system is shown in figure 7. It consists of a stiffened, cantilever steel beam (E), one end of which is mounted onto an aluminium frame (B), which is forced by an electromagnetic shaker (A). Near the free end of the beam, a pair of permanent/electromagnet stacks (F) is mounted on the frame. The electromagnets are powered by a computer-controlled power supply (Agilent E3647A). The relative oscillations of the beam are measured by a pair of laser vibrometers (D). A small notch (C) in the beam is machined between the stiffeners and clamped end. This notch is used to initiate fatigue crack formation and propagation.

This experimental system allows for the introduction of three slow-time processes. One directly related to damage is initiated by a stress concentration caused by the notch in the beam. The other two are provided by supplying variable voltages to the electromagnets, and these voltages provide the variable operating conditions. For this paper, it was decided to alter the supply voltages sinusoidally, so that the voltage phase portrait resembled a figure of eight. Thus, the following voltage signals are supplied to the electromagnets: $v_1 = 10 + 9\sin(t)$ and $v_2 = 10 + 9\cos(0.5t)$, where t is measured in hours.

Owing to the beam stiffeners, the vibrations of the beam are constrained mainly to one mode. The frequency response functions obtained for the small oscillations in each energy well show that the dynamic bandwidth of the system is well under 50 Hz, with main resonances occurring below 10 Hz. Therefore, the signals from the laser vibrometers are passed through a low-pass filter with a 50 Hz cut-off frequency before digitization. Total change in the natural frequency of small oscillations in each of the energy wells due to voltage drop from 19 to 1 V was of the order of 4%, translating into approximately 8% decrease in the stiffness of the beam.

Signals from the laser vibrometers and both electromagnet open circuit voltages were recorded with 160 Hz sampling frequency employing an NI PCI-6052E, 16 bit A/D data acquisition card. The system was forced sinusoidally with 10 Hz frequency and amplitude of the input was adjusted so that the oscillations exhibited nominally chaotic behaviour. The beam fractured at approximately 12.5 hours into the experiment, and a total of approximately 7 million data points were acquired for each channel.

The selection of chaotic motions, as a starting point in the experiment, was motivated by two practical reasons. First, the reference model built on the chaotic data is expected to be valid in a large region of the fast-time phase space; alternatively one can use stochastic interrogation (Cusumano & Kimble 1995) to construct the equivalent reference model. Second, a chaotic system with drifting parameters makes it more likely to observe bifurcations in fast-time behaviour, which allows one to observe the robustness of this method to bifurcation noise.

5. Results

The first 2^{15} points of vibration data are used to reconstruct a phase space for the reference model construction. Standard techniques are used to determine the embedding dimension of 5 and delay time of 6 sampling times. A simple local linear model based on 16 nearest neighbours is used to estimate $\boldsymbol{P}(\boldsymbol{y};\boldsymbol{\phi}_\mathrm{R})$ for equation (3.2). The collected data are split into a total of 1728 disjoint data records with 2^{12} data points in each. The reference reconstructed phase space is partitioned into 81 hypercuboids, so that each hypercuboid contains approximately the same number of points. Based on this partition, the 1728×405 tracking matrix $\boldsymbol{Y}$ is assembled. The randomly chosen columns of this matrix are shown in figure 8. There are discernable trends in the traces that correspond to the sinusoidal variations and power-law type fatigue damage accumulation. However, they are largely obscured by the noise.

To check for the existence of the voltage subspace, the whole tracking matrix $\boldsymbol{Y}$ is projected in the least-squares sense onto the power supply voltage states recorded separately. That is, the measured, mean subtracted voltage is arranged into a matrix $\boldsymbol{V} \in \mathbb{R}^{1728\times 2}$; a projection operator is determined as $\boldsymbol{B} = (\boldsymbol{Y}^\mathrm{T}\boldsymbol{Y})^{-1}\boldsymbol{Y}^\mathrm{T}\boldsymbol{V}$; and the voltage subspace is obtained as $\boldsymbol{P} = \boldsymbol{Y}\boldsymbol{B}$. The original phase portrait of the voltage is shown in figure 9*a* and the

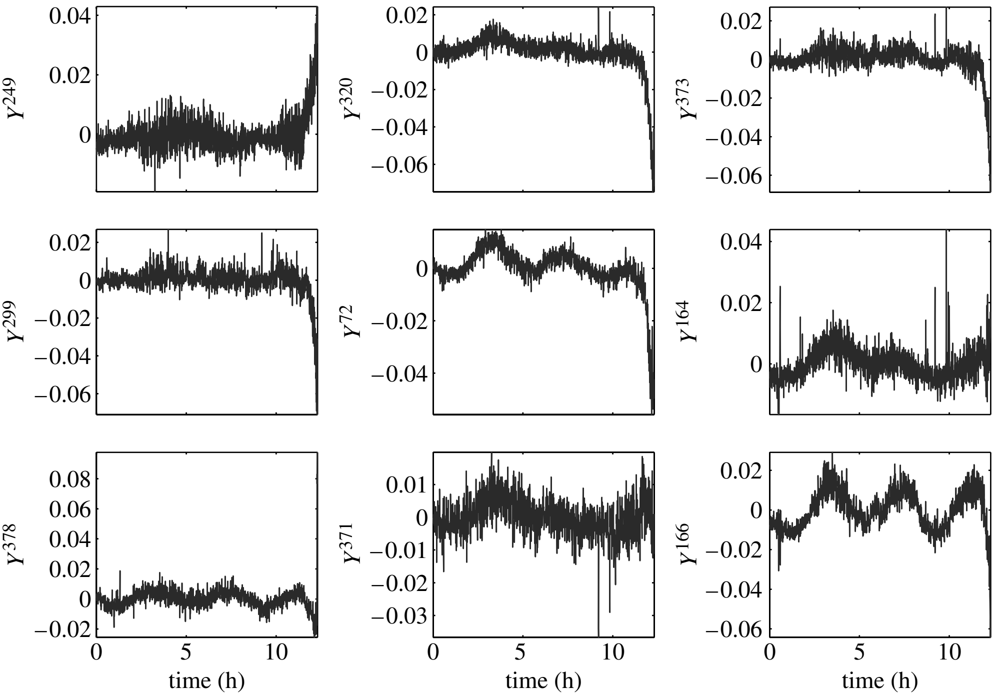

Figure 8. Randomly chosen columns of the tracking matrix $\boldsymbol{Y}$ versus time.

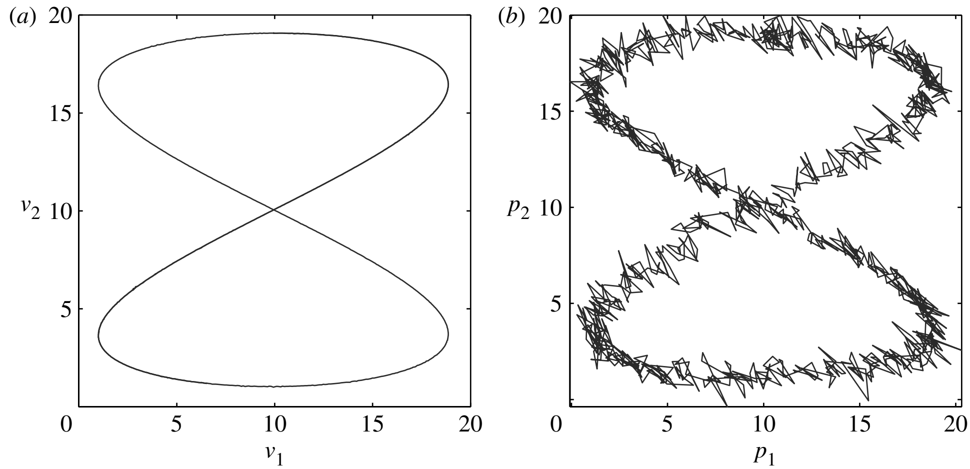

Figure 9. (*a*) The original voltage phase portrait and (*b*) the corresponding phase portrait obtained from the tracking matrix.

corresponding projection of the tracking matrix is shown in figure 9*b*. The topological similarity of these two plots is apparent, despite the large local fluctuations observed in the projection. This noise can be substantially reduced by using larger data records for estimating the feature vectors, or by filtering the results. However, since this is not the focus of our study, all the remaining results will be presented as raw and unfiltered.

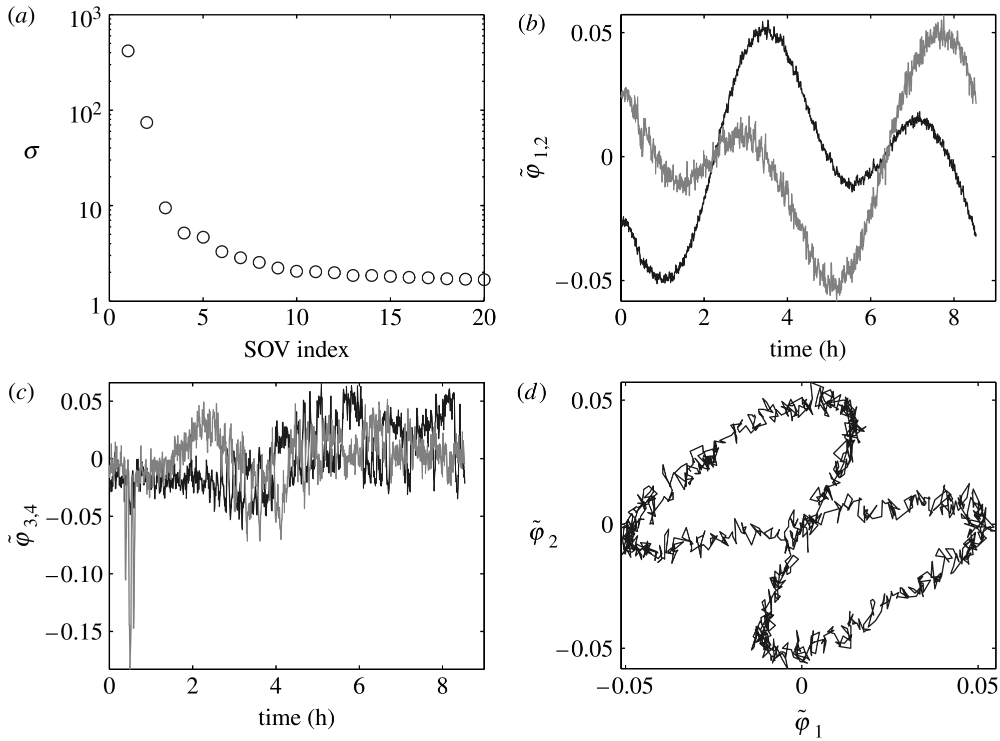

Figure 10. Blind identification of the voltage-related feature subspace: (*a*) first 20 SOVs of the first 1200 rows of the tracking matrix; (*b*) first two SOCs corresponding to the largest two SOVs; (*c*) next two SOCs; and (*d*) phase portrait corresponding to the first two SOCs.

The first 1200 rows of the tracking matrix are used to blindly identify the SOMs and POMs related to the changes in the voltages shown in figure 9. Figure 10*a* shows the SOVs of this data, which clearly indicates two dominant singular values that are much larger than the rest. The corresponding SOMs are depicted in figure 10*b*,*c*. Note that the first two of these in figure 10*b* are much smoother and have a larger overall variation than the following two coordinates, which are hard to be distinguished from the original columns of $\boldsymbol{Y}$ shown in figure 8. The phase portrait obtained using these first two SOCs is shown in figure 10*d*.

The tracking matrix is projected onto modes that are not related to the operating conditions (i.e. modes corresponding to $\tilde{\boldsymbol{\varphi}}_1$ and $\tilde{\boldsymbol{\varphi}}_2$ are omitted) to obtain a new tracking matrix $\bar{\boldsymbol{Y}}$. The SOD analysis of this matrix shows that there is only one SOV that is substantially larger than the rest, as shown in figure 11*a*. The corresponding SOC is shown in figure 11*b*, and the plots in figure 11*c*–*f* show the following four SOCs. It is clear from these plots that the remaining plots are dominated by noise, but they are still sensitive to a dramatic failure at the end of the experiment.

The first SOC shown in figure 11 is normalized by scaling it to vary from 0 to 1. This result is shown in figure 12*a* and its reciprocal is shown in figure 12*b*. In Chelidze & Cusumano (2006), a similar system with an unperturbed potential field is studied to identify the fatigue damage. The results obtained for that paper are very similar to those of figure 12. As in that paper, here the reciprocal of the identified trend shows a linear decay with respect to time, as predicted by the Paris law of fatigue crack growth for carbon steel materials.

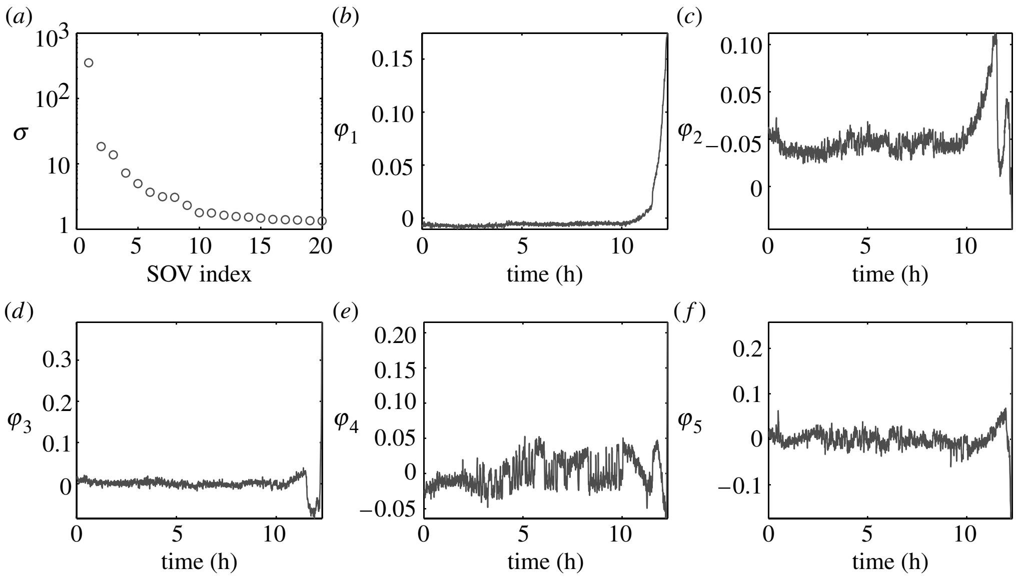

Figure 11. Fatigue damage identification: (*a*) largest 20 SOVs of the truncated tracking matrix; (*b–f*) the five SOCs corresponding to the largest five SOVs.

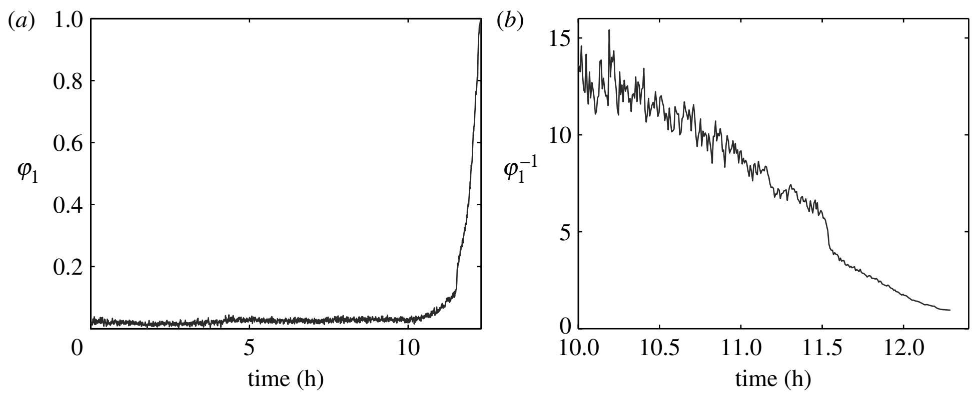

Figure 12. (*a*) The dominant SOC of the truncated tracking matrix and (*b*) its reciprocal versus time.

6. Discussion

The individual components of the tracking matrix show a large amount of local fluctuations or noise. However, some of the general trends of both voltage signals are also discernible in the traces as shown in figure 8. Many of the components also have large variations at the end of the experiment right before the fracture in the beam. Some abrupt jumps are also present in the data that correlate well with the observed bifurcations in the fast-time dynamic behaviour.

The existence of a plane (two-dimensional subspace) of the battery voltage in the feature vector space is demonstrated by projecting the whole tracking matrix onto the actual battery voltage data. The results (figure 9) clearly demonstrate that such a plane exists in the feature space, but it is contaminated by a substantial amount of noise. This noise is not constrained only to this plane, but extends into the whole feature space. This leakage might complicate the unique identification of this subspace.

In practice, the operating conditions subspace needs to be blindly identified using the portion of the tracking matrix that does not experience damage accumulation. In our case, we can assume that at the beginning of the experiment there is no substantial change in the fatigue damage state. Using the corresponding data, figure 10*b*,*d* clearly shows that the first two SOCs trace out a trajectory that corresponds to the actual voltage phase portrait. Thus, the corresponding SOMs span the battery variation subspace identified in figure 9. In addition, the SOVs shown in figure 10*a* are strong indicators that only two independent sources of variation are present in the data; the first two SOVs are at least one order of magnitude larger than the rest.

The following pair of the SOCs shown in figure 10*c* exhibits some variations that cannot just be attributed to noise, which can impede an accurate identification of the voltage plane. Part of this variation, characterized by sudden jumps, is caused by bifurcations in the fast-time behaviour of the system. If the actual voltage variation subspace is not constrained to a plane but a surface, then this variation can also be present in the remaining SOCs. Furthermore, if the time scales of different variations are similar, then unique blind separation of the corresponding subspaces may not be possible. However, in this case, the first two SOMs still provide an accurate approximation of the voltage surface: the angle between the subspaces span by the projecting vectors of figure 9 and the corresponding first two SOMs of figure 10 is very close to 0 (0.0730 rad, as calculated using MatLab command 'SUBSPACE'). Thus, the SOD can accurately identify the subspace span by the changes in the voltage, even in the presence of substantial experimental variability.

Finally, our hypothesis is validated by applying the SOD to the feature subspace that does not contain the SOMs that correspond to the voltage variations. The SOD results shown in figure 11 clearly demonstrate that there is only one scalar slow-time process present in this subspace; the first SOV is at least one order of magnitude larger than the rest. This result is in complete agreement with our previous study (Chelidze & Liu 2005). The corresponding SOC also shows similar power-law type trend. The remaining SOCs are either discontinuous or contain mainly noise. The next two SOCs show high sensitivity to the dramatic changes at the end of the experiment, but have clear discontinuities and considerably more noise. As in Chelidze & Liu (2005), the beam was made of carbon steel. The Paris law of macroscopic crack growth for this material stipulates a quadratic form of slow-time equation in equation (3.1). This translates into linear decay in time for the reciprocal of the damage state (crack length in the Paris law). The results shown in figure 12 clearly confirm this stipulation as was the case for Chelidze & Liu (2005), where the corresponding data were used for a successful failure prediction.

This technology can clearly be applied to situations where the damage evolution is not prominent at the beginning of system monitoring. This time period is usually expected to be of considerable length after the initial break-in period. If this is true, all variations in operating or environmental conditions can be identified prior to damage identification. Note that this methodology can also filter out all the operating condition changes that are abrupt. However, it will probably fail in situations when the operating condition changes both abruptly and smoothly. In this case, the corresponding subspace needs to be identified by some other means, like PCA or independent component analysis.

7. Conclusion

A general framework for the reconstruction of slow-time deterministic (damage) dynamics using fast-time (vibration) measurements from a dynamical system under variable operating conditions is described. The fast-time vibration data are assumed to be quasi-stationary over the time period of each observation. The PSW feature vectors are constructed by quantifying warping (deformation) of the reconstructed fast-time phase space for each of these observations. During the reference (healthy) period of operation, SOD is used to identify the subspace of the feature space affected by the time-varying operating conditions. The damage trajectory is then reconstructed in the complement of the feature space using SOD. The theory is validated experimentally using a beam, subjected to fatigue damage, vibrating in a time-varying nonlinear potential field.

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