Problem Solution: A Structured Approach

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The current approaches of the problems solved in textbooks are a mixture of intuitive approach, analytical tactic, a priori knowledge of the required quantities to proceed to the solution (that a student is missing), mixing numbers and variables in equations, numbers with unit not in the base form, with results appearing as base or derived units, etc. In this paper we propose a methodological approach that will eliminate memorization, will develop organized thinking, and provide a structured approach to the solution process of problems, especially as seen in physics and the various engineering disciplines. The symbolic notation is followed in the solution process. The symbolic notation provides inside to the relationship between the independent variable and the dependent variables, if any, something the numerical substitution will never reveal. The question of how to start solving the problem is proposed to be simply "by answering the question". The result is a final relationship that to the left of the equal sign has the unknown quantity and to the right has all known quantities. The numerical answer is provided in three steps the symbolic final formula, substitution of the numerical values with units, and final answer with the units, if any. This approach has the advantage that all problems are solved following the same structured approach, avoiding the notion that every problem is its own case.

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Introduction/Background

It has been said that Engineers solve problems. Problem solving is part of the Engineering education. But how do we develop problems? Where do we learn from to solve problems? Nowhere the problem and its solution process are studied in the course curriculum. Various approaches can be found in the examples (solved problems) presented in the various textbooks. Many include guidelines in the end of the chapters for specific type of problems [1, 2, 3, 4, 5]. Other textbooks try to generalize the approach of solving problems, mostly from the mathematical point of view [6, 7]. Others try to create acronyms of the method. (SOLVEM method) [8]. There are books devoted to specific categories of problems like physics [9, 10, 11]. Finally, there are many collections of solved problems in Physics [12, 13, 14].

Data/Formulation/Methodology

A structured approach is organized, proposed and presented that promotes the idea of a uniform approach to all problems solution process as, at least, appear in an introductory level, algebra and calculus based physics course. The problems are approached following the same tactic. Here we shall refer to problems in physics for which the approach was developed. It is a methodological general approach to solve problems in physics and engineering. Typically, it is required to:

Study the problem.

Understand what phenomenon or phenomena are involved.

Understand what is given.

Understand what is asked to be determined.

Consider appropriate assumptions.

At this point the method splits in two approaches, the analytic and the synthetic.

- Analytic approach:

In this type of problems: we are dealing with one phenomenon and an equation exists or can be arranged to provide to the left the unknown quantity and to the right known quantities. The typical question is "Determine an expression for…" while the typical answer is "In order to determine the …" In this case, we write down the equation that has the unknown quantity to the left. We continue substituting the unknown quantities to the right with expressions that eventually will result in a final expression in which to the left is the unknown quantity and to the right all quantities are known.

- Synthetic approach:

In this type of problems: there is not a single equation that gives the unknown quantity in terms of known quantities. In this case, because we are dealing with more than one phenomena and there is not a single equation that can be arranged to provide to the left the unknown quantity and to the right known quantities the unknown quantity must me found by equations describing the system. This kind of problems is typically the case of simultaneous phenomena. The typical question is "Determine an expression for…" while the typical answer is "In order to determine the... we must find… and …" In this case, we write down an equation that includes the unknown quantity due to one phenomenon and write one or more additional equations that include the unknown quantity due to the other phenomenon or phenomena. We form a system of appropriate number of equations that can be solved for the unknown quantity. At the end, we have an expressions in which to the left is the unknown quantity and to the right all known quantities.

Analysis

The approach has been presented, tested, and further developed in class for the past two years. The students are initially surprised of the approach. Additionally, the students are unfamiliar and amazed with the use of symbolic notation to the development of the solution of a problem. After solving a few examples the students start seeing the pattern and appreciate the method. The year before the presentation of the unified structural method, the problems were solved as in the textbook. The disadvantage was that every problem was its own case. Given that every week we cover a different chapter or topic, it soon became obvious that the students were memorizing than thinking. The proposed approach leaves out memorization and promotes thinking.

As an example we shall consider the solution of typical problems found in most textbooks of Calculus based Physics. Although the various textbooks present a variety of approaches for these problems, we present a unified approach that starts from the same principle and proceeds in order to arrive to the desired result.

- Analytic Approach:

In this type of problems we start from an expression that contains the unknown quantity and substitute other expressions until everything to the right of the equation sign is known.

The following problems, provided for illustration of the method, use Gauss law to determine the electric field everywhere. The law in electricity, states that the electric flux over a closed surface is proportional to the enclosed net electric charge. Mathematically this statement is written as

$$\Phi_E = \iint_S E \cdot dS = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \qquad (1)$$

Notice that the steps to get the solution are identical although the problems represent different cases.

Practice 1: Electric field of a point electric charge: Assume a point electric charge Q determine an expression for the electric field everywhere.

Typical textbook: Takes the answer ready from Coulomb's law and uses the result to prove the law.

Proposed strategy: In order to find the electric field we shall use Gauss law and solve for the electric field.

Because the source of the electric field is a point charge, we shall use a spherical coordinate system to solve the problem because one of the coordinate surfaces, the sphere has the same distance from the point charge. We shall set the point charge to be the origin of the spherical coordinate system in order to simplify calculations.

For the given configuration we apply Gauss law:

$$\iint_S E \cdot ds = \frac{Q_{encl}}{\varepsilon_0} \qquad (2)$$

In this equation we are looking to find the electric field, E is the unknown quantity. Its general form, in spherical coordinates, is:

$$E = E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi} \qquad (3)$$

The element of surface ds is known, it is the element of surface on the Gaussian sphere of radius r. The element of surface of the sphere in spherical coordinates is:

$$ds = r \sin\theta d\theta d\phi \, \hat{r} \qquad (4)$$

The charge Q is known.

The permittivity of the material, free space, is also known.

Substituting (3) and (4) into (2) we get:

$$\iint_S \left(E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi} \right) \cdot \left(\sin\theta d\theta d\phi \, \hat{r} \right) = \frac{Q_{encl}}{\varepsilon_0} \qquad (5)$$

Performing the indicating operations we get:

$$\int_0^{2\pi} \int_0^{\pi} E_r r \sin\theta d\theta d\phi = \frac{Q_{encl}}{\varepsilon_0}$$
$$\therefore E_r = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2} \qquad (6)$$

The solution, the electric field has only radial component, and we can write the solution in vector form as

$$E = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2} \hat{r} \quad r > 0 \qquad (7)$$

Practice 2: Electric field of a charged conducting sphere: Assume electric charge Q on the sphere. Determine an expression for the electric field everywhere.

Proposed strategy: In order to find the electric field we shall use Gauss law and solve for the electric field.

Because the source of the electric field the electric charge, is a distributed on a conductive sphere, it is distributed on the surface of the sphere. We shall use a spherical coordinate system to solve the problem because one of the coordinate surfaces, the sphere, has the same distance from the surface distributed electric charge. We shall set the center of the charged sphere to coincide with the spherical coordinate system in order to simplify calculations.

For the given configuration we recognize two material regions: that of the conductive sphere and that of the free space. We need to apply Gauss law twice, once for each region.

For the region inside the sphere the accumulated electric charge is zero.

By Gauss law,

$$\iint_S E \cdot ds = \frac{Q_{encl}}{\varepsilon_0} \tag{8}$$

The right hand of (8) is zero ($Q_{\text{enclosed}} = 0$); the right hand has the element of surface on the Gaussian surface which is not zero because the Gaussian sphere radius R>0. Then the electric field has to be zero,

$$E = 0 \quad r < R \tag{9}$$

For the region outside the sphere, the enclosed electric charge is Q. Appling Gauss law,

$$\iint_S E \cdot ds = \frac{Q_{encl}}{\varepsilon_0} \tag{10}$$

In this equation we are looking to find the electric field, E is the unknown quantity. Its general form, in spherical coordinates, is:

$$E = E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi} \tag{11}$$

The element of surface ds is known, it is the element of surface on the Gaussian sphere of radius r. The element of surface of the sphere in spherical coordinates is:

$$ds = r \sin\theta d\theta d\phi \, \hat{r} \tag{12}$$

The charge Q is known.

The permittivity of the material, free space, is also known.

Substituting (11) and (12) into (10) we get

$$\iint_S \left(E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi} \right) \cdot \left(\sin\theta d\theta d\phi \, \hat{r} \right) = \frac{Q_{encl}}{\varepsilon_0} \tag{13}$$

Performing the indicating operations we get:

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi} E_r r \sin\theta d\theta d\phi = \frac{Q_{encl}}{\varepsilon_0} \\ &\therefore E_r = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2} \end{aligned} \tag{14}$$

The solution, the electric field has only radial component, and we can write the solution in vector form as

$$E = \frac{Q_{encl}}{4\pi\varepsilon_0 r^2} \hat{r} \quad r > R \tag{15}$$

And the general solution for the whole space is:

$$E = \begin{cases} 0 & r < R \\ \dfrac{Q_{encl}}{4\pi\varepsilon_0 r^2} \hat{r} & r > R \end{cases} \tag{16}$$

Practice 3: Electric field of an infinitely long thin wire: Assume electric charge Q on the sphere. Determine an expression for the electric field everywhere.

Proposed strategy: In order to find the electric field we shall use Gauss law and solve for the electric field.

Because the source of the electric field the electric charge, is a distributed along an infinitely long wire, it is distributed with line charge density λ along the wire. We shall use a circular cylindrical coordinate system to solve the problem because one of the coordinate surfaces, the cylinder, has the same distance from the line distributed electric charge. We shall set the axis of the circular cylinder to coincide with the circular cylindrical coordinate system in order to simplify calculations.

For the given configuration we recognize one material regions: that of the free space. We need to apply Gauss law once.

By Gauss law,

$$\iint_S E \cdot ds = \frac{Q_{encl}}{\varepsilon_0} \tag{17}$$

In this equation we are looking to find the electric field, E is the unknown quantity. Its general form, in circular cylindrical coordinates, is:

$$E = E_\rho \hat{\rho} + E_\phi \hat{\phi} + E_z \hat{z} \tag{18}$$

The element of surface ds is known, it is the element of surface on the Gaussian circular cylinder of radius ρ. The closed cylindrical surface consists of three open surfaces; these of the two bases of the cylinder and thar of the circular cylinder.

The elements of surface of the two bases are:

$$\begin{aligned} dS_B &= \rho d\rho d\phi \left(-\hat{z} \right) \\ dS_T &= \rho d\rho d\phi \left(\hat{z} \right) \end{aligned} \tag{19}$$

The net flux contribution of both surfaces is zero.

The element of surface of the circular cylinder is

$$ds = \rho d\phi dz \hat{\rho} \tag{20}$$

The charge Q is known.

The permittivity of the material, free space, is also known.

Substituting (18) and (20) into (17) we get

$$\iint_S \left(E_\rho \hat{\rho} + E_\phi \hat{\phi} + E_z \hat{z} \right) \cdot \left(\rho d\phi dz \hat{\rho} \right) = \frac{Q_{encl}}{\varepsilon_0} \tag{21}$$

Performing the indicating operations we get:

$$\begin{aligned} &\int_0^{L} \int_0^{\pi} E_\rho \rho d\phi dz = \frac{Q_{encl}}{\varepsilon_0} \\ &\therefore E_r = \frac{Q_{encl}}{4\pi\varepsilon_0 \rho^2} \end{aligned} \tag{22}$$

And the general solution for the whole space is:

$$E = \frac{Q_{encl}}{4\pi\varepsilon_0 \rho^2}\hat{\rho} \quad \rho > 0 \tag{23}$$

A second set of examples is presented from the topic of electric potential. The problems are stated and the uniform approach of solution is emphasized. Three relations are considered as the basis to solve the problems.

Electric potential due to a collection of discrete charges distribution:

$$V = \frac{1}{4\pi\varepsilon}\sum_i \frac{Q_i}{r} \tag{24}$$

Electric potential due to a collection of continuous charge distribution:

$$V = \frac{1}{4\pi\varepsilon}\int \frac{dq}{r} \tag{25}$$

and, electric potential difference due to the electric field:

$$V_a - V_b = -\int E \cdot dl \tag{26}$$

Practice 1. An electric dipole consists of discrete point charges +Q and –Q. determine the electric potential at a point in the space.

Proposed solution: The electric potential at any point will be the sum of the potential due to charge Q1 and the potential due to charge Q2 (superposition principle). The potential V1 due to charge Q1 is known as well as the potential V2 due to charge Q2. Substituting, we get the net potential.

$$\left.\begin{array}{l} V = V_1 + V_2 \\ V_1 = \frac{1}{4\pi\varepsilon}\frac{Q_1}{r_1} \\ V_2 = \frac{1}{4\pi\varepsilon}\frac{Q_2}{r_2} \end{array}\right\} V = \frac{1}{4\pi\varepsilon}\sum_1^2 \frac{Q_i}{r_i} = \frac{1}{4\pi\varepsilon}\left(\frac{Q_1}{r_1} - \frac{Q_2}{r_2}\right) \tag{27}$$

Practice 2. Consider a non-conducting rod of length L. The rod has electric charge Q uniformly distributed along it. Determine an expression for the electric potential at any point located on the line perpendicular to the rod and passing through its midpoint (the bisector line to the rod).

Proposed solution: The potential at a point along the bisector is the sum of the elementary potentials due to elementary charges along the rod.

$$V = \int dV \tag{28}$$

The process continues until all quantities to the right have been expressed in terms of known quantities:

The elementary potential, dV, due to an elementary charge, dq, has been found earlier.

The elementary charge, dq, is found in terms of the charge (given) and the geometry (given).

The distance from the source to the field point is found and expressed in terms of the geometry of the system.

Finally, substituting back, we get the expression for the potential.

$$dV = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{dq}{R}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} dq = \lambda d \\ d_{\ x} = dx \end{array}\right\} dq = \lambda dx \\ \lambda = \frac{Q}{L} \end{array}\right\} dq = \frac{Q}{L}dx \tag{29}$$

$$R = \sqrt{x'^2 + y^2}$$

$$dV = \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{\lambda dx}{\left(x'^2 + y^2\right)^{1/2}} \tag{30}$$

$$V = \int_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{\lambda dx}{\left(x'^2 + y^2\right)^{1/2}} \tag{31}$$

Performing the integration either using a computer Algebra System (CAS) or an analytical approach using calculus, we het the final expression for the potential at any point along the bisector line in terms of the given quantities of the system:

$$\begin{aligned} V &= \int_{-L/2}^{L/2} \frac{1}{4\pi\varepsilon_0\varepsilon_r}\frac{\lambda dx}{\left(x'^2 + y^2\right)^{1/2}} \\ &= \frac{\lambda}{4\pi\varepsilon_0\varepsilon_r}\ln\left[x' + \sqrt{x'^2 + y^2}\right]\Bigg|_{-L/2}^{L/2} \\ &= \frac{\lambda}{4\pi\varepsilon_0\varepsilon_r}\ln\left[\frac{\left(\frac{L}{2}\right) + \sqrt{\left(\frac{L}{2}\right)^2 + y^2}}{-\left(\frac{L}{2}\right) + \sqrt{\left(\frac{L}{2}\right)^2 + y^2}}\right] \end{aligned} \tag{32}$$

- <u>Synthetic approach:</u>

Practice: An automobile is speeding in a school zone. A police car is parent just outside the school. The police car start from rest and accelerates with constant acceleration a. Determine an expression for the time is takes for the polish car to catch the speeder?

Proposed strategy: Assumption: We assume that both automobiles travel in one dimension path. Key observation: Both automobiles travel the same distance from the moment the police car starts moving to the time it catches the speeder,

$$s_P = s_A \tag{33}$$

The police car travels uniformly accelerated motion with constant acceleration, distance,

$$s_P = v_0 t + \frac{1}{2}at^2 \tag{34}$$

While the speeding automobile travels with constant velocity, distance

$$s_A = v_A t \tag{35}$$

Substituting (34) and (35) into (33) we get

$$v_0 t + \frac{1}{2}at^2 = v_A t \tag{36}$$

Solving (36) for t and noting that $v_0 = 0$, we get

$$t = \frac{2v_i}{a} \tag{37}$$

- Three step evaluation of a formula or calculation of the arithmetic value of a quantity:

A symbolic approach to the solution process of a problem has been utilized to determine the solution of the problem. A major advantage of this approach is the clear appearance of the dependent variable on the independent variables direct or inverse. Furthermore, the problem is solved once and the solution of the problem for every arithmetic value can be calculated. In practical applications we need to determine a numerical value for the answer.

The three steps include:

1) formula,

2) substitution of numerical values with units, and

3) final answer that includes numerical value and units, if they exist.

As an example:

The final velocity of an electric charge accelerated in an electric field created by potential difference, is:

$$v = \sqrt{\frac{2q_0\left(V_a - V_b\right)}{m}} = \sqrt{\frac{2\cdot 2\times10^{-9}\mathrm{C}\cdot\left(1350\mathrm{V} - (-1350\mathrm{V})\right)}{5\times10^{-9}\mathrm{kg}}} = 46\mathrm{m/s} \tag{38}$$

The process can me standardized using a mathematical notepad [15] and make the evaluation of the dependent variable fast for various numerical values of the independent variables, Figure 1.

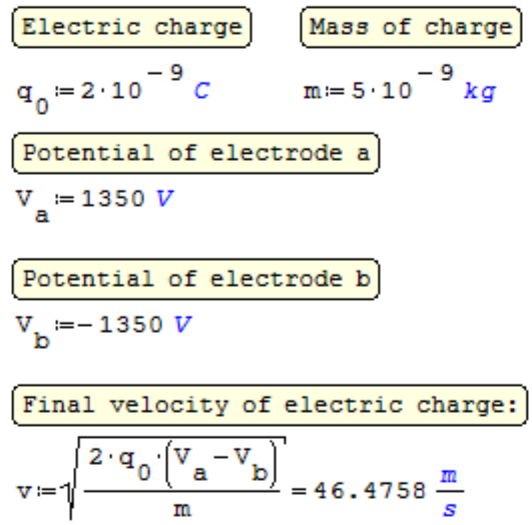

Figure 1. Mathematical notepad used for numerical evaluation.

Conclusions

We have presented a methodological and unified approach to solve problems and specifically problems in general physics and Engineering disciplines. The proposed method has been presented to first year calculus based physics successfully. The method is applied either as analytical, if one phenomenon is involved, or as synthetic, if two or more phenomena are involved. The two approaches, analytic and synthetic, can be applied in order to provide, through logically guided steps, the final expression with the unknown quantity to the left and everything known to the right. The numerical calculation to the general symbolic solution may follow.

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